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Research under this grant was conducted in identification and control of distributed parameter systems, particularly damping mechanisms in distributed elastic systems, modeling of flexible structures, control of systems with nonlinear behavior and control of systems with solitary waves. A thorough analysis if damping mechanisms in elastic systems was carried out, including common classical forms of damping, and led to a new model of damping("spatial hysteresis") which correctly models observed asymptotic behavior in the frequency domain. More recent work focused on transfer function methods for infinite dimensional linear systems. Five PhD students were supported and completed their dissertations under this grant. Nineteen papers were published during this period, including "Some remarks on transfer function methods for infinite dimensional linear systems", "Spectral and asymptotic properties of linear elastic systems with internal damping", and "frequency/period estimation and adaptive rejection of periodic disturbances." (Lcd/b

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FINAL SCIENTIFIC REPORT ON RESEARCH SUPPORTED BY AFOSR 85 - 0263

by David L. Russell

Principal Investigator

UW MIPAC Coordinator

Period Covered: September 30, 1985 through September 29, 1988

1. <u>General Description of the Research Program</u>. The subject grant provided funds supporting research in a number of areas during the period indicated. These include: control and identification if distributed parameter systems, damping mechanisms in distributed elastic systems, modelling of large flexible structures, control of systems exhibiting nonlinear, self-excited oscillations and control of nonlinear wave systems involving solitary waves. Funds were used to support research activity by the principal investigator, visiting senior faculty researchers and graduate research assistants, and also short term consultants and visitors. In addition to salary support, funds were used to support scientific computing relevant to the research program, domestic and foreign travel by the principal investigator and one of his research assistants, and to purchase needed supplies and equipment related to operation of the UW MIPAC Facility.

Interim Scientific Reports detailing activities during the first two years of the grant period have been filed earlier. We adjoin copies of these reports as appendices to this report. In the main body of the present report we detail scientific activities carried out during the third and final year of the period, from September 30, 1987, to September 29, 1988.

2. Principal Results of the Research Program.

The main accomplishments of the research program supported by the subject grant can be summarized as follows. First of all, a total of five doctoral students were graduated during or shortly after the grant period, all partially supported by the grant.

Graduating PhD Students Supported by the Program.

Dr. Katherine Kime (now Asst. Prof. at Case Western Res. U.)

Dr. Thomas Svobodny (Post Doc at Virginia Tech)

Dr. Robert Acar (Post Doc at Univ. of Oklahoma)

Dr. Khosro Shabtaie (current status unknown)

Dr. Scott W. Hansen (Post Doc at Virginia Tech)

As a result of the sponsored scientific research activity noted above or carried out in previous years under earlier AFOSR grants, a number of journal articles, reports and dissertations appeared under UW MIPAC auspices during the reporting period.

Publications, Theses.

1. D. L. Russell: *Frequency / Period Estimation and Adaptive Rejection of Periodic Disturbances*, SIAM Journal on Control and Optimization, 34 (1986), pp. 1276-1308.

2. D. L. Russell: *A Floquet Decomposition for Volterra Equations with Periodic Kernel and a Transform Approach to Linear Recursion Equations*, Journal of Differential Equations, 68 (1987), pp. 41-71.

3. D. L. Russell: *On the Positive Square Root of the Fourth Derivative Operator*, Quarterly of Applied Mathematics 46, December, 1988.

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4. D. L. Russell: *Some remarks on transfer function methods for infinite dimensional linear systems,* submitted to SIAM Journal on Control.

5. D. L. Russell: *On mathematical models for the elastic beam with frequency proportional damping,* to appear in SIAM "Frontiers of Applied Mathematics" volume edited by H. T. Banks, probably in 1989.

6. D. L. Russell: *Mathematical models for the elastic beam and their control-theoretic implications,* in "Semi-groups, theory and applications, Vol. II", #152 in Pitman "Research Notes in Mathematics", Longman House, Harlow, UK, 1986, pp. 177 - 216.

7. D. L. Russell: *Spectral and asymptotic properties of linear elastic systems with internal damping,* to appear in Proc. Conf. on Boundary Stabilization and Control, Clermont-Ferrand, France, 1987, J. Simon, Ed.

8. G. Chen, S. G. Krantz, D. L. Russell, C. E. Wayne, H. H. West and M. P. Coleman: *Analysis, design and behavior of dissipative joints for coupled beams,* to appear.

9. Katherine A. Kime: *Boundary Controllability of Maxwell's Equations in a Spherical Region,* Thesis, University of Wisconsin, Madison, August, 1986.

10. Thomas J. Bridges: *On the Two-Dimensional Periodic Surface Waves Occurring in Rectangular Vessels: Theory Versus Experiment,* Technical Summary Report #2878, Mathematics Research Center, University of Wisconsin-Madison.

11. Thomas P. Svobodny: *State reconstruction and dynamic observers for nonlinear differential equations,* Thesis, University of Wisconsin - Madison, August, 1987.

12. Thomas P. Svobodny and D. L. Russell: *Phase identification in linear time-periodic systems,* to appear in IEEE Transactions on Automatic Control.

13. **Thomas** P. Svobodny: *Stability of nonlinear observers for dissipative ordinary differential equations*, to appear in Int. J. Control.

14. Robert Acar: *Identification of Coefficients in Elliptic Systems*, Thesis, University of Wisconsin - Madison, August, 1987 (Submitted to SIAM J. Cont. & Opt.)

15. Khosro Shabtaie: *State and Parameter Identification in Elastic Systems*, Thesis, University of Wisconsin - Madison, August, 1987.

16. Scott W. Hansen: *Frequency-proportional damping models for the Euler-Bernoulli beam equation*, Thesis, University of Wisconsin - Madison, December, 1988

17. W. Hereman, P. P. Banerjee and D. Faker: *Construction of solitary wave solutions of the Korteweg-de Vriess equation via Painlevé analysis*, CMS Technical Summary Report #89-8, Center for the Mathematical Sciences, University of Wisconsin - Madison, September 1988

18. P. P. Banerjee, W. Choe, G. Cao and W. Hereman: *Stationary eigenmodes and their stability during wave propagation in a medium with quadratic and cubic nonlinearities without dispersion*, CMS Technical Summary Report #89-9, Center for the Mathematical Sciences, University of Wisconsin - Madison, September, 1988

19. W. Hereman, P. P. Banerjee and M. R. Chatterjee: *Derivation and implicit solution of the Harry - Dym equation, and its connections with the Korteweg - de Vriess equation*, to appear.

Report on Scientific Activities Supported by AFOSR 85 - 0263
During the Period September 30, 1987, to September 29, 1988.

I. Overview

During the indicated period the principal investigator, visiting senior research faculty and graduate research assistants connected with the UW MIPAC Facility carried forward a program of research and experimentation in various areas of control theory and related aspects of applied mathematics. Areas of particular emphasis in this research program include:

(a) Control and stability of linear distributed parameter systems, transform methods for input/output admissibility and closed loop system analysis and description.

(b) Structural damping mechanisms in distributed elastic systems;

(c) Control theory of nonlinear partial differential equations exhibiting solitary wave solutions.

These activities were pursued by the principal investigator, three visiting senior research faculty, i.e.,

Prof. Willy Hereman, Van Vleck Assistant Professor, University of Wisconsin - Madison (this is a visiting position); active during entire report period.

Prof. Partha P. Banerjee, Syracuse University; Spring semester, 1988 (also supported in part by the Department of Electrical and Computer Engineering)

Prof. Gunter Leugering, Technische Hochschule, Darmstadt, West Germany; October, 1987.

Prof. Jack Carr, Heriot - Watt University, Edinburgh; partial support during spring semester, 1988.

The **services** of a number of short term consultants were supported, including:

> Prof. G. Chen, Texas A. & M. University,
> Prof. Ruth Curtain, University of Groningen,
> Prof. Elena Fernandez, Virginia Tech.,
> Prof. Luther White, University of Oklahoma.
> Dr. K. D. Graham, Honeywell, Inc., Minneapolis

Equipment consulting services provided by MTS Corp., Minneapolis, SMS Corp., Detroit, and Hewlett - Packard, Rolling Meadows, Illinois, all in connection with MIPAC Facility equipment, were purchased with grant funds.

In the next section we describe in more detail the research areas listed as (a),(b) and (c) above, indicating the role played in these researches by the individuals indicated here.

In the final section of the report we describe the *Workshop on Computational and Experimental Aspects of Control*, convened in Madison during May of 1988 with partial support from AFOSR, and we briefly describe other aspects of the research program, such as participation in scientific meetings, foreign and domestic travel, etc.

II. **Further Description of the Research Program**.

Here we discuss in somewhat greater detail the particular
areas of research emphasis listed in the preceding section.

(a) Control and stability of distributed parameter systems,
transform methods for input/output admissibility and closed
loop system analysis and description.

This effort has consisted primarily in two parts. The first
part concerns the controllability of elastic beam models which
incorporate dissipation terms of such a nature as to cause the
model to exhibit frequency proportional, or asymptotically fre-
quency proportional, modal damping behavior. In this research
the principal investigator has worked closely with a graduate
research assistant, Scott W. Hansen, who received the PhD degree
in Mathematics from the University of Wisconsin in December of
1988, shortly after the end of the grant period. Mr. Hansen
received partial support from the subject grant which materially
assisted him in his research program. The thesis deals with so-
called *bending rate* damping, which in the Euler - Bernoulli con-
text is modelled by the partial differential equation

$$\rho\frac{\partial^2 w}{\partial t^2} - 2\gamma\,\frac{\partial^3 w}{\partial t\partial x^2} + EI\,\frac{\partial^4 w}{\partial x^4} = 0,$$

along with modified boundary conditions to *ensure energy dissipa-
tion*, and with what has been called the *spatial hysteresis* model

$$\rho\frac{\partial^2 w}{\partial t^2} - 2\gamma\,\frac{\partial}{\partial x}\int_{x-\delta}^{x+\delta} h(x-\xi)\left[\frac{\partial^2 w}{\partial t\partial x}(x,t) - \frac{\partial^2 w}{\partial t\partial x}(\xi,t)\right] + EI\,\frac{\partial^4 w}{\partial x^4} = 0,$$

also equipped with appropriate boundary conditions. The thesis
involved examination of natural boundary conditions, estimates of
eigenvalues and eigenfunctions, comparison with the so-called
square root damping model

$$\rho\,\ddot{w} + 2\gamma\,A^{1/2}\,\dot{w} + A\,w = 0,$$

analysis of controllability with distributed and point actuators,
perturbation and resolvent methods to establish holomorphicity
of the related strongly continuous semigroup in the energy space,
and other related topics.

Stability studies were carried out by Prof. Gunter Leugering
of Technische Hochschule Darmstadt, who visited during October,
1988. He has been concerned with certain damping mechanisms in
elastic structures which are of viscoelastic type, involving
time delay, or memory, terms. The asymptotic stability proper-
ties of a number of models of this sort were studied using modi-
fied Liapounov methods. Prof. Leugering was supported in part by
the subject grant during his visit.

The 1980's have witnessed a resurgence of research interest in
the transfer function description of linear input - output systems.
These methods have a number of attractive properties, particularly
in regard to questions of robustness with respect to system varia-
tions, as developed, e.g., in the papers of Doyle and Stein, as well
as many others, too numerous to cite in any representative way.

The application of these methods, sometimes called H^{∞} methods be-
cause of the boundedness property in a right half plane character-
istic of transfer functions of well-posed, nonanticipative linear
systems, has been extended to infinite dimensional ("distributed
parameter") systems only relatively recently. We cite the works of
Callier and Desoer, Curtain and Pritchard, Salamon, Francis among
others. Numerous references are cited in the well known text by
Francis and in a recent expository article by R. Curtain.

The majority of "H^{∞}" studies for infinite dimensional systems
have been conducted in the context of systems possessing transfer
functions in the so-called *Callier - Desoer* class, which excludes
transfer functions important in a variety of applications, such as
undamped, or lightly damped, elastic systems described by partial
differential equations of hyperbolic type, neutral functional equa-
tions, etc.

Our recent research does not treat H^{∞} control synthesis methods;
rather, it aims to develop certain properties of transfer functions
of some systems which do not belong to the Callier - Desoer class. It
turns out that transfer functions of systems such as those we have
treated have some rather unusual, one might even say disconcerting,

properties, particularly in relation to convergence questions with
respect to approximation via finite dimensional systems. In addition
to these approximation questions, we have studied the use of transfer
function methods to analyze closed - loop systems arising out of cer-
tain linear feedback laws, use of transfer functions to determine the
admissibility of input and output mechanisms, and a number of other,
related, matters. In the process we have developed a representation
of the semigroup in terms of its inverse Laplace transform which,
in its specific context, is less restrictive than the resolvent int-
egral representation in Dunford and Schwartz.

The present work is, for convenience, couched in the single in-
put - single output framework but most of the results obtained extend
rather directly to systems with a finite number of inputs and out-
puts. This is certainly not the case for H^∞ methods in general be-
cause of the non-commutativity of matrix multiplication, but that
noncommutativity plays a minor role relative to the topics of inte-
rest to us in our work so far, which is reported in detail in the
paper (4.) listed earlier in this report.

The primary motivation for this research may be summarized as
follows. The recent success of frequency domain methods virtually
guarantees that a large fraction of future distributed parameter con-
trol research will be carried out within this framework. That grant-
ed, it seems important to us that the connections between frequency
domain representations and their state space counterparts should be
fully explored, with appropriate cautions posted where those connec-
tions appear to be somewhat tenuous. We are hopeful that the work
described here will represent a useful step in that direction. A
logical direction for future work is to develop H^∞ synthesis tech-
niques for systems of the sort just described, including co-prime
factorization techniques, etc., taking into account the properties
actually possessed by systems related to important application areas.
A copy of the paper in which this research is documented is attach-
ed as Appendix C, below.(a)

(b) Structural_damping_mechanisms_in_distributed_elastic_systems.

In this research we continued our efforts toward finding mo-
dels replicating observed damping phenomena in simple elastic

systems, the Euler and Timoshenko beam equations being used as prime examples. A prime consideration was to obtain models based on physical principles and thus defensible from that point of view as well as from that of mathematical rigor and convenience. Two mechanisms new in this context were introduced; namely, *thermoelastic* damping and *shear diffusion* damping. These mechanisms are *indirect* in the sense that they involve the coupling of the mechanical equations governing beam motion to related dissipative systems with their own dynamics, resulting in an overall system in which mechanical energy is dissipated.

Thermoelastic damping involves the heat diffusion process

$$\frac{\partial T}{\partial t}(x,t) = - k\, T(x,t) + K\, \frac{\partial^3 w}{\partial t \partial x^2}(x,t)$$

coupled with the familiar Euler - Bernoulli equation

$$\rho\, \frac{\partial^2 w}{\partial t^2} + EI\, \frac{\partial^4 w}{\partial x^4} + L\, \frac{\partial^2 T}{\partial x^2} = 0$$

by means of the w term in the first equation and the T term in the second. The complete system consists of these two equations The complete system then consists of the two indicated equations with appropriate boundary conditions, which we do not elaborate upon here. The damping action is explained in terms of differential heating of opposite sides of the beam in bending, with subsequent heat conduction, resulting in a loss of mechanical energy.

We can replace the pair of equations by a single equation by differentiating the first equation twice with respect to x and substituting into the equation obtained from the second by applying to it the operator $\frac{\partial}{\partial t} + k\, I$, arriving finally at

$$\rho\, \frac{\partial^3 w}{\partial t^3} + \rho k\, \frac{\partial^2 w}{\partial t^2} + (EI+ LK)\, \frac{\partial^5 w}{\partial t \partial x^4} + EIk\, \frac{\partial^4 w}{\partial x^4} = 0 \ .$$

Working with this equation we can see that the exponential damping rate versus frequency relationship is quadratic at low fre-

quencies, tending to asymptotically constant rates as the frequency tends to infinity. This behavior is suspected to be accurate for certain metallic beams in which thermoelastic effects might be expected to be significant contributors to the overall vibration damping effect.

The shear diffusion damping model begins with the Timoshenko beam model which, written in terms of the lateral displacement w and the shear angle β, becomes

$$\rho\frac{\partial^2 w}{\partial t^2} - I_\rho\left[\frac{\partial^3\beta}{\partial t^2\partial x} + \frac{\partial^2 w}{\partial t^2\partial x^2}\right] + EI\left[\frac{\partial^3\beta}{\partial x^3} + \frac{\partial^4 w}{\partial x^4}\right] = 0 \ ,$$

$$I_\rho\left[\frac{\partial^2\beta}{\partial t^2} + \frac{\partial^3 w}{\partial t^2\partial x}\right] + K\beta - EI\left[\frac{\partial^2\beta}{\partial x^2} + \frac{\partial^3 w}{\partial x^3}\right] = 0 \ .$$

In the second equation all terms except the first can be regarded as *shearing* forces to which the shear angle β responds through the action of the first term. In the shear diffusion model we suppose that a further viscous force affects the evolution of β:

$$I_\rho\left[\frac{\partial^2\beta}{\partial t^2} + \frac{\partial^3 w}{\partial t^2\partial x}\right] + 2\sigma\ \frac{\partial\beta}{\partial t} + K\beta - EI\left[\frac{\partial^2\beta}{\partial x^2} + \frac{\partial^3 w}{\partial x^3}\right] = 0 \ .$$

Most commonly we assume I_ρ to be very small relative to the other constants present and we neglect the first term. What remains is a diffusion process for β, hence the name. After some manipulation we eventually arrive at the single equation

$$\frac{2\sigma}{K}\ \frac{\partial}{\partial t}\left[\rho\ \frac{\partial^2 w}{\partial t^2} - \frac{\rho EI}{2\sigma}\ \frac{\partial^3 w}{\partial t\partial x^2} + EI\ \frac{\partial^4 w}{\partial x^4}\ \right] + \rho\ \frac{\partial^2 w}{\partial t^2} + EI\ \frac{\partial^4 w}{\partial x^4} = 0.$$

It turns out that here the damping versus frequency relationship is cubic at low frequencies, tending asymptotically to a linear relationship at high frequencies, asymptotically in agreemenmt with many experimental studies of internal damping. By combining thermoelastic and shear diffusion damping in different ratios, a variety of behaviors corresponding to those observed under various circumstances in the laboratory can be obtained.

The research reported under (a), carried out by Research

Assistant Scott W. Hansen, insofar as it deals with mathematical
properties of partial differential equations modelling dissipa-
tion in elastic beams, is also in important component of the
research area being described here.

(c) Control theory_of nonlinear partial differential_equations
    exhibiting_solitary wave_solutions.

This research has been carried out in cooperation with Re-
search Associate, Dr. Willy Hereman, Visiting Assistant Professor
Dr. Partha P. Banerjee (Syracuse U.) and Research Assistant, Mr.
Zhang Bing - Yu, for whom a portion of this research will consti-
tute his doctoral dissertation. It concerns partial differential
equations such as the Korteweg - de Vriess (KdV) equation

$$\frac{\partial w}{\partial t} + 3 w \frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3} = 0 \ ,$$

other nonlinear partial differential equations, such as the Bous-
sinesq, modified KdV, and Harry - Dym equations which, like the
KdV equation, exhibit solitary travelling wave solutions, and au-
xiliary linear equations used in conjunction with these, e.g.,

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} = 0$$

and other third order linear equations are useful in studying the
KdV equation.

Professors Hereman and Banerjee have been working on the re-
lationships between the various partial differential equations
which exhibit solitary waves. With the aid of MACSYMA, rather
remarkable transformations have been discovered mapping, e.g.,
the KdV equation over into the Harry - Dym equation, at first
appearing to be quite dissimilar to the KdV equation. In fact,
a group of transformations connecting KdV, modified KdV and the
Harry - Dym equation has been discovered. These results are
significant because in control and other studies on these equa-
tions, which find applications in fluid dynamics, optical signal
transmission, neuron pulses, and many other areas, it will be

possible to concentrate studies on a single prototype model, probably the KdV model since its properties are well documented in the literature.

Research Assistant Mr. Zhang Bing - Yu has been studying the control properties of the KdV equation with a boundary control term. Results are still preliminary, but promising. Linearized equations, which are of third order, have been investigated by a variety of methods, including Laplace transform methods, and have been seen to have controllability properties not very different from the heat equation. Mr. Zhang is continuing his work under a successor AFOSR grant.

III. Other Activities Supported by the Grant.

Travel supported by the grant during the subject period included:

i.   A trip by the Principal Investigator to the December, 1987 IEEE/SIAM CDC Conference in Los Angeles, where a paper on dissipation in elastic systems was presented.

ii. A trip by the Principal Investigator to Detroit, Michigan, to confer with personnel of Structural Measurements, Inc., on operator of MIPAc Facility software.

iii. A trip by Associate Investigator, Dr. Willy Hereman, to the 1988 Summer SIAM meeting in Minneapolis.

iv. (As a result of a decision to carry out 1988-89 academic year operations at Virginia Tech.) A trip by the Principal Investigator to Blacksburg, Virginia to discuss and carry out, in part, transfer operations.

During the period May 16, 17, 18, 1988, a *Workshop on Experimental and Computational Aspects of Control* was convened in Madison to review recent developments and techniques in this area. Several speakers presented talks, including H. T. Banks, J. A. Burns, P. P. Banerjee, J. McLaughlin, L. White, A. Laub, J. S. Gibson, W. Littman. While the majority of workshop expenses were covered by an NSF grant, we are happy to acknowledge the partial support of the subject AFOSR grant in helping to pay transportation costs and other expenses of speakers and participants, as well as partial salary support for the organizers during the period of planning for the workshop. We are particularly gratified that Lt. Col. James Crowley of AFOSR was able to be present at the Workshop and take part in its activities.

APPENDIX A

Report on Activities Supported by the Grant During Year 1.

UW MIPAC

Department of Mathematics
University of Wisconsin
Madison, Wisconsin
53706

MODELLING, INFORMATION PROCESSING, AND CONTROL   December 31, 1986

INTERIM SCIENTIFIC REPORT ON RESEARCH SUPPORTED BY AFOSR 85 - 0263

by David L. Russell

Principal Investigator

UW MIPAC Coordinator

Period Covered: September 30, 1985 through September 29, 1986

1. **General**.   The subject grant supported research in control and identification of distributed parameter systems, large flexible structures, and nonlinear self-excited oscillations during the period September 30, 1985 through September 29, 1986.   Funds were used to support research activity by the Principal Investigator, several graduate Research Assistants and visiting senior faculty from other institutions who were supported here for short periods as consultants in connection with various projects.   In addition to salary and visitor support, funds were used to support scientific computing relevant to the research program, to support domestic and foreign travel by the Principal Investigator and one of his Research Assistants, and to purchase needed supplies and equipment related to operation of the UW MIPAC Facility in two locations on the UW Madison Campus.   Below we describe these research activities in greater detail and then adjoin appendices consisting of several recent research reports.

2. **Research Activities**. During the subject grant period the Principal Investigator, Research Assistants and Visitors connected with the UW MIPAC Facility carried forward a program of research and experimentation in various areas of mathematical systems theory. Areas of particular emphasis in this research include:

(a) Theory and implementation of distributed parameter systems control;

(b) Parameter identification for distributed systems of elliptic and parabolic types;

(c) Modelling, control and state estimation for nonlinear systems with concentration on systems exhibiting self-excited oscillations and those subject to periodic forcing.

(d) Control of wave processes governed by the Maxwell electromagnetic equations.

(e) Development of mathemtical models for structural damping mechanisms in elastic structures.

These activities were pursued by the Principal Investigator, a total of five Research Assistants, including:

> Robert Acar,
> Scott Hansen,
> Katherine Kime,
> Khosro Shabtaie,
> Thomas Svobodny,

and by at least eight visiting Senior Consultants and other short term visitors from other institutions, including

S. Banda, Wright-Patterson Air Force Base,
Thomas Bridges, Worcester Polytechnic Institute,
G. Chen, Pennsylvania State University,
Joyce O"Halloran, University of Wisconsin - Milwaukee,
Willy Hereman, University of Ghent, Belgium,
D. L. Lukes, University of Virginia,
Luther White, University of Oklahoma,
Robert Wheeler, Virginia Polytechnic Institute.

The visit of Dr. S. Banda of Wright-Patterson AFB was extremely helpful in opening up channels of communication with that facility and the Air Force Institute of Technology in particular. While he was on our campus, Dr. Banda gave a talk on robust control research projects currently under way at Wright-Patterson.

Below we comment in somewhat greater detail concerning the research areas listed above and we indicate the role played in furthering this research by individual Research Assistants, Consultants and Visitors.

In connection with areas (a)and (d) we are particularly gratified to report completion of the PhD requirements by Katherine Kime, who graduated in August, 1986. Subsequently, "Kathy" accepted a rare opportunity granted by the government of France, in the form of the *Bourse Chateaubriand*, which provided funds for her to spend a post-doctoral year in France working with Prof. A. Blaquiere and his group and with the Control Group at INRIA in Versailles. It is anticipated that she will accept a tenure track, or further post-doctoral, position in the United States at the end of her visit to France. Her receipt of the *Bourse Chateaubriand* is quite noteworthy as the total number of awards in the U.S. each year (not restricted to mathematics) is on the order of twenty or so.

Kathy's thesis, an excerpt from which is attached as Appendix A, is concerned with the control of electromagnetic fields in three dimensional spatial regions by means of controlling currents flowing on the boundary of that region. Earlier research carried out by the Principal Investigator, and supported by AFOSR, established field controllability in an infinite cylindrical region under the assumption that the fields and applied control currents

were independent of the axial coordinate of the cylinder. In her
thesis Kathy extended this work to a cylinder of finite length,
the fields and controls being subject to periodic boundary condi-
tions at the ends of the cylinder and then went on to the far more
difficult task of analyzing the control problem in regions of arbi-
trary geometry. This work is particularly intricate because of the
necessity to study divergence-free solutions of the vector wave
equation, including the specification of the very special Hilbert
spaces, related to divergence-free solutions, which serve as state
spaces for these electromagnetic processes. The correct framework
for posing the control problem was developed in a general geometric
contextg and specific affirmative control results were obtained by
moment-theoretic methods in a three dimensional ball-shaped region.
It is expected that parts of this work will appear in the SIAM Jour-
nal on Control and Optimization and in the Proceedings (A) of the
Royal Society of Edinburgh.

Research in area (e), the modelling of structural damping mech-
anisms, advanced significantly during the subject reporting period.
The integro-partial differential equation model for structural damp-
ing described in the final scientific report on AFOSR 84 - 0088 one
year ago has been related, at least partially, to a physical damping
mechanism developed by Dr. Clarence Zener, a prominent physical and
engineering theoretician associated with Carnegie-Mellon, and desc-
ribed in his well known text *Elasticity and Anelasticity.* This, of
course, does not mean that Zener anticipated our mathematical model.
Rather, it means that our mathematical model is consistent with his
physical description, which is given in terms of thermoelastic ef-
fects and heat conduction within the elastic beam being studied.
These connections were discovered, almost accidentally, during a
visit to Carnegie-Mellon by the Principal Investigator.

The prototype mathematical model for structural damping is the
"square root" model described the the Principal Investigator and G.
Chen of Pennsylvania State University in the *Quarterly of Applied
Mathematics,* 1974. The integro-partial differential equation mod-
el referred to in the previous paragraph is a rather natural out-
growth of the work which was done to develop the square root model
which, abstractly, takes the form of a second order system in Hil-
bert space of the form

$$\rho \frac{d^2 w}{dt^2} + 2\gamma A^{1/2} \frac{dw}{dt} + A w = 0 ,$$

where $A^{1/2}$ denotes the positive square root of the positive self-adjoint elasticity operator A. While mathematically appealing, this model, applied to the elastic beam case, can be assigned a definite physical interpretation in terms of damping forces proportional to local bending rates only in those cases where the boundary conditions on A, essentially the fourth order derivative operator for the Euler - Bernoulli beam, are such that the positive square root $A^{1/2}$ coincides with the negative second derivative operator. Consequently, we have felt it to be desirable to explore the mathematical relationship between the negative second derivative operator and $A^{1/2}$ for those boundary conditions where the two do not coincide. Work largely carried out during the subject reporting period has shown that, with D denoting the negative second derivative operator, we have

$$D = (I + P) A^{1/2}$$

where P is a bounded, but in general not compact, operator related to the classical Muntz-Szasz theory of real exponentials. As a result it is seen that D has domain including that of $A^{1/2}$ in all cases. Further work is directed toward determining in what sense, if any, the operator P may be regarded as small in order to facilitate an appropriate perturbation theory.

Related work in this area is being carried out by Research Assistant Scott Hansen, who is exploring the structure of the vibrational modes for a variety of systems serving as structural damping models. This work has been well advanced during the period and is expected to lead, eventually, to Mr. Hansen's PhD dissertation.

Research area (b), parameter estimation for elliptic systems, has received considerable attention during the period. The Principal Investigator and Research Assistant Robert Acar attended the University of Oklahoma Symposium, largely devoted to this topic, during October, 1985, the former presenting a paper at the meeting concerning results obtained with an equation error method based on linear programming which shows considerable promise. Robert Acar

has been developing, and is currently completing a PhD dissertation
on, an equation error method for computing an unknown coefficient
function p(x,y) in the elliptic equation

$$\nabla \cdot p \nabla \varphi = f .$$

Mr. Acar's method uses a duality method in order to avoid having to
actually compute approximations to the Laplacian of the solution φ.
All that is required is that the data should admit reasonably reli-
able estimation of the gradient of φ, $\nabla\varphi$. Initial computational
tests of the method have proved encouraging and further tests of the
method are planned in the course of the completion of the thesis.
The visit to this campus of Professor Luther White of the University
of Oklahoma in August, in conjunction with the first MIPAC workshop
has resulted in continuing and significant contacts between Mr. Acar
and Professor White in this intriguing and difficult area.

Our efforts in research area (c), modelling, control and state
estimation for nonlinear systems, are being carried out by the Prin-
cipal Investigator and Research Assistant Thomas Svobodny. The lat-
ter is expected to complete a PhD dissertation on this subject and
graduate in June, 1987. The main interest focusses on a nonlinear
differential system which includes a two dimensional component whose
uncoupled representation takes the form

$$\frac{d^2 y}{dt^2} + g(y,\frac{dy}{dt}) \frac{dy}{dt} + h(y) = f(t) .$$

In the unforced case, $f(t) \equiv 0$, the main interest centers on self-
excited oscillations - periodic solutions with a minimum period T
which arises due to differing stability properties near the origin
as compared with the far field. Such oscillations occur in many
physical contexts, such as wing and fuselage flutter, panel flutter,
power plants under lean fuel conditions, etc. - far too many appli-
cations to do justice to here. We are primarily concerned with the

question of being able to identify the state y, $\frac{dy}{dt}$, along with the
state vector, x, of a coupled elastic system during operation of
the coupled system, usually on the basis of measurement of a limit-
ed number of components of the elastic state vector, x. In the
course of his work, Mr. Svobodny has developed methods originally

designed for state estimation of periodic linear systems so that
they are now effective for use in the neighborhood of a periodic
solution of a nonlinear system such as we have exhibited here. We
have also carried out some work in connection with the forced sys-
tem, wherein f(t) is a periodic forcing function affecting the sys-
tem. Mr. Svobodny and the Principal Investigator were very much
the beneficiaries of the visit to the UW Campus of Professor D. L.
Lukes of the University of Virginia, which began in September of
1986 and which was partially supported by the subject AFOSR grant.
We feel it is also significant that some of the data used by Mr.
Svobodny in his work has been obtained from real aerodynamic sys-
tems with the aid of UW MIPAC equipment, acquired with the partial
support of AFOSR under the Universities' Instrumentation Program.

Other work in the nonlinear area has been carried out by Dr.
Thomas Bridges, a visiting Assistant Professor, now with Worcester
Polytechnic Institute. Dr. Bridges carried out extensive studies
of bifurcations of sloshing waves in a tank subject to a periodic
disturbance, comparing theoretical calculations with observations
made on a water-filled tank excited with UW MIPAC's Bruel & Kjaer
Vibration Exciter. Substantial agreement between theory and ex-
periment was recorded and documented in a report prepared by Dr.
Bridges at the end of his research.

Finally we report on work covered partly by research area (b)
above and by research area (a). The Principal Investigator and
Research Assistant Khosro Shabtaie have been studying identification
problems associated with the Euler - Bernoulli beam equation,

$$\rho \, \frac{\partial^2 w}{\partial t^2} + EI \, \frac{\partial^4 w}{\partial x^4} = 0,$$

which may be regarded as being of parabolic type since the family
of characteristics is degenerate, along with similar equations
which incorporate various structural damping mechanisms, as descri-
bed earlier under (e). Shabtaie has been investigating on-line
parameter identification schemes, in particular a method newly dev-
eloped at UW MIPAC which we call the estimator predictor method.
It has some features in common with Y. Landau's model reference
method but appears to outperform it quite significantly on tests.
These tests have been carried out using real data obtained from a
beam in the UW MIPAC Analysis unit, the data being collected and

analyzed with our HP 5451C System Analyzer from a steel beam set into vibration with the Bruel and Kjaer electromagnetic shaker.

In this area the Principal Investigator has recently been interested in identification methods based on frequency domain methods, including methods which rely entirely on the measured natural frequencies of the above-displayed system to determine $EI(x)/\rho(x)$ and methods, now proving very interesting, for estimation of this quantity from the trace of the transfer function of the system on the imaginary axis of the complex plane, the function most easily observed using standard FFT techniques. The transfer function identification techniques, in addition to showing promise for use in applications, are highly interesting from the mathematical point of view because of their relationship to Hilbert and Hankel transforms We expect to present a paper on this subject at the forthcoming International Conference on Industrial and Applied Mathematics in Paris, scheduled for June 29 - July 3, 1987.

3. Supporting Activities.

(a) Publications. As a result of the scientific research activity described above or carried on in previous years under other AFOSR grants, a number of journal articles, reports and dissertations appeared under UW MIPAC auspices during the reporting period. These include the following:

1. D. L. Russell: *"Frequency / Period Estimation and Adaptive Rejection of Periodic Disturbances."* Appeared in SIAM Journal on Control, Vol. 34 (1986), pp. 1276-1308. Reprint enclosed with this report.

2. D. L. Russell: *"A Floquet Decomposition for Volterra Equations with Periodic Kernel and a Transform Approach to Linear Recursion Equations."* Submitted to, and accepted by, the Journal of Differential Equations.

3. D. L. Russell: *"On the Positive Square Root of the Fourth Derivative Operator."* Prepared as UW MIPAC Report. Now being readied for journal submission, probably to the Quarterly of Applied Mathematics. A copy is attached as Appendix A.

4. Katherine A. Kime: *"Boundary Controllability of Maxwell's Equations in a Spherical Region".* Thesis, University of Wisconsin, Madison, August, 1986. Appeared in part as Mathematics Research Center Technical Summary Report, enclosed as Appendix B.

5. Thomas J. Bridges: *"On the Two-Dimensional Periodic Surface Waves Occurring in Rectangular Vessels: Theory Versus Experiment."* Mathematics Research Center Technical Summary Report #2878. A copy is attached as Appendix C.

Additionally, some of the work of Prof. Dahlard L. Lukes, who visited here over a period partially included in the reporting period, is being written up as a MIPAC Report. It will be forwarded to AFOSR when available. Professor Goong Chen of Pennsylvania State University is preparing a report on segmented elastic beams which, in part, describes results on natural frequency distribution

of such **beams** obtained with the use of UW MIPAC facilities.

(b) Educational. In addition to the standard educational activities
inherent in graduate education, thesis supervision, etc., UW MIPAC
is pursuing the idea of a Workshop on Mathematical Modelling and
Control Implementation. It is our hope to be able to offer this
workshop at least biennially, beginning in the academic year 1987-
1988. A proposal for partial NSF support to supplement our AFOSR
funds to this end has been submitted to the Applied Mathematics Div-
ision of NSF. A prototype workshop, to determine initial guidelines
for the operation of such a workshop was held at UW MIPAC in August,
1986. A series of five talks was presented by the Principal Inves-
tigator in the general area of Fast Fourier Transform analysis, with
emphasis on natural frequency identification, damping rate estima-
tion, recognition of nonlinear dynamics, and other topics in the
general area of mathematical modelling. Initial results were prom-
ising but did indicate definite problem areas which are currently
under study. Very careful attention to experiment preparation and
modes of display of graphic materials was strongly indicated by
prototype workshop experience.

(c) Travel. In May and June, 1986, the Principal Investigator trav-
elled to Pakistan, partially supported by AFOSR funds, visiting and
giving lectures at Karachi University, Quaid-e-Azam University in
Islamabad, the capital of Pakistan, and Bahauddin Zakkariya Univer-
sity of Multan. The reception of the eight lectures presented was
most enthusiastic and significant scientific contacts were made.
The Principal Investigator was impressed with the scientific efforts
being made at the institutions visited, in spite of generally small
nationwide numbers of scientific personnel and severe deprivation
in the area of modern computing facilities. Some efforts to allev-
iate the latter situation were evident during the visit to Multan
where a new DEC VAX 1130 was being installed.

(d) UW_MIPAC_Facility Equipment Augmentation_and_Research Plans.

As a result of a $220,000 grant under the 1986-1987 DoD Uni-
versities Instrumentation Program, UW MIPAC is installing addition-
al experimental equipment in its Model Development Unit at 1307

University Ave. in Madison. The new equipment includes a Hewlett-Packard 3526 Fourier Analyzer, a HP 310 minicomputer and related peripheral equipment, a DISA Laser Vibrometer and an MTS High Frequency Vibration Test Facility. The first two are already in place, delivery of the Laser Vibrometer is expected within the next few weeks, and we expect the High Frequency Vibration Test Facility (HFVTF henceforth) to be installed during spring 1987. The HFVTF will enable us to study fairly large and complex flexible structures both in passive vibration and in the presence of external disturbances and controls. The Laser Vibrometer will, for the first time, enable us to make non-contact measurements on models used for experiment. We will be making the facilities of UW MIPAC widely available to other researchers, both on this campus and elsewhere. Professor Goong Chen of Penn State has already used our facilities for segmented beam studies, Professor Robert Wheeler of VPI has checked some properties of beams with viscoelastic damping, and Professor Luther White of the University of Oklahoma will be visiting in March, 1987 to carry out experiments connected with elastic modulus coefficient identification for aluminum plates. The new DISA Laser Vibrometer and the MTS HFVTF will be used by Professor White to carry out scanning measurements of aluminum plates in order to obtain data for identification computations connected with algorithms which he has been studying as part of his research program.

APPENDIX B

Report on Activities Supported by the Grant During Year 2

UW MIPAC

Department of Mathematics
University of Wisconsin
Madison, Wisconsin
53706

MODELLING, INFORMATION PROCESSING, AND CONTROL

Technical Report on Activities Supported by

AFOSR Grant 85 - 0263

during the period

September 30, 1986 through September 29, 1987

submitted by

David L. Russell

UW MIPAC Coordinator

Principal Investigator

1.   **General**.

        During the period September 30, 1986, through September 29,
1987, which is the second year of operation of UW MIPAC under fund-
ing provided by the subject grant (hereinafter we will refer to
"the second grant year" to describe this period) a large variety
of scientific activities were undertaken with partial or complete
support from the subject grant. These activities include:

  i) Research Pursued by the Principal Investigator;

 ii) Research carried out by associates at the University of Wis-
     consin or by visitors to UW supported in part by the subject
     grant;

iii) Research carried out by research assistants and dissertators
     supported in part by the subject grant; PhD graduates.

 iv) Experimental Research carried out in the UW MIPAC Laboratory;

  v) Joint experimental research carried out with UW MIPAC visitors
     partially supported by the subject grant;

 vi) Installation and testing of new UW MIPAC equipment;

vii) Seminars, invited speakers;

viii) Documentation and bibliographical efforts;

 ix) Scientific travel, lectures;

        We proceed now to report in more detail on the activities car-
ried out during the second grant year in each of these categories.

i) **Research Pursued by the Principal Investigator**

During the second grant year the Principal Investigator has been concerned with several areas of research.

A continuing interest has been that of damping mechanisms in elastic structures; elastic beams in particular. While more realistic models have been suggested in the interim, the original journal article by the Principal Investigator and Prof. G. Chen, now Professor at Texas A & M University, which first set frequency proportional damping on a firm mathematical footing, remains widely quoted. Its main disadvantage has been that, for a wide variety of elasticity operators A, the operator coefficient $A^{1/2}$ in the equation

$$x'' + 2\gamma A^{1/2} x' + A x = 0$$

has remained a poorly understood mathematical entity. To a degree that mystery has been removed with the completion of the Principal Investigator's paper on the positive square root of the fourth order operator

$$A w = \frac{d^4 w}{dx^4}$$

in the general case which includes all "natural" boundary conditions. In this paper it is shown that in all of these cases

$$A^{1/2} = (I + Q) D^2 , \quad D^2 w = \frac{d^2 w}{dx^2} ,$$

where Q is a bounded, but in general not compact, operator. In the case of "paired" boundary conditions, where the eigenfunctions of A are purely trigonometric, D = 0. The proof of the main result just indicated shows that $A^{1/2}$ is always relatively bounded with respect to D^2 but is not generally relatively compact with respect to that operator. The companion result to the effect that

$$D^2 = (I + P) A^{1/2} ,$$

where P is also a bounded operator, is also proved. In addition to

mathematical interest, these results are also important because they impact the theory of other models, such as what has been called the "spatial hysteresis model", to the degree that these other models are perturbations of the basic square root model. These results will appear in the *Quarterly of Applied Mathematics*.

Because experience with UW MIPAC Facility equipment has indicated that information on the *transfer function*, or *frequency response function*, is a readily available indicator of many distributed parameter system characteristics, the Principal Investigator, in his most recent work, has turned to a study of transfer functions of distributed parameter systems and the uses to which they can be put. A paper presented at the *International Congress on Industrial and Applied Mathematics* (ICIAM) in Paris, France, July, 1987, demonstrates that the values of the transfer function are often mathematically sufficient in order to determine the coefficients of the elasticity operator - this is true, for example, in the case of a non-uniform string and a non-uniform elastic beam. Other cases are under study. The results also indicate, via the classical *identity theorem* of complex analysis, that any finite frequency segment of the transfer function is, in principle, sufficient to identify the coefficients. Naturally, the problem of recovering the complete coefficient function from a short transfer function segment is highly ill-conditioned. Efforts are underway to investigate the computational implications of these results.

Because of the increasingly evident importance of the transfer function in distributed parameter control analysis, and because the identification results just referred to make essential use of transfer function properties as they relate to linear feedback in distributed parameter systems, it has also seemed advisable to re-study some of the properties of transfer functions in the context of distributed parameter systems, and particularly with reference to unbounded input elements, such as occur in the case of boundary value control. We have not attempted to match the erudition in Salamon's epic paper but, rather, we have focussed on transfer function properties which significantly affect potential distributed parameter applications, such as indications for admissible inputs and admissible feedback relations, as well as modal and finite element approximations. A significant part of the work has been the elucidation of an apparent paradox recently brought out in correspondence between Prof. Ruth Curtain of the University of Groningen in the Netherlands and Prof. G. Chen of Texas A & M University.

Finally, we have begun to study certain problems connected with the controlled initiation of solitary waves in certain nonlinear wave processes. A significant part of this work has been the development of a new, convolution-based, numerical algorithm which allows solution of these nonlinear equations in a computationally economic way so as to permit development of interesting system behavior on readily available microcomputers, such as the IBM PC / AT. We will have more to say about this research in later sections of this report.

ii) <u>Research Carried out by Associates at the University of Wisconsin or by Visitors to UW Supported in Part by the Subject Grant</u>;

The first part of this section is a short report on research carried out by Prof. Willy Hereman, who has been supported, in part, as a research associate under this grant. We were very fortunate to have Professor Hereman join our group in the fall of 1987. He is working in nonlinear and solitary wave theory, and in acousto-optics. In the first area referred to, he has been assisting the Principal Investigator in supervision of a student's PhD. thesis, which address a problem of optimal control for nonlinear wave systems. His skills in the second area are particularly pertinent because of their relationship to laser vibrometer measurement processes, discussed further in § iv).

Professor Hereman's research work here in the UW MIPAC program has been primarily concerned with the study of direct solution methods for nonlinear partial differential equations which describe solitary wave propagation in various physical situations. The methods studied, which are algebraic in character, apply to single equations, such as those of Korteweg - de Vries or sine - Gordon type, and also to systems of coupled equations of related types.

In collaboration with Prof. P. Banerjee of Syracuse University Electrical Engineering Dept., Prof. Hereman has been investigating the remarkable connections between the Harry - Dym, KdV and modified KdV equations. The nonlinear transformations linking all of these have been discovered and efforts are ongoing to derive new solutions based on the properties of the transformations. The final goal is

to give an overall heuristic guide for the modelling and solution of nonlinear dispersive systems, using a variety of direct approaches. These efforts are to be summarized in a forthcoming SIAM Review article now in preparation.

Professor Hereman, and also Professor M. Slemrod (supported by another AFOSR grant, not reported here) are working with a graduate student, Mr. B.-Y. Zhang, in a study of certain optimal control problems associated with nonlinear equations exhibiting soliton solutions. The Principal Investigator has also been involved in this project through his work in developing numerical solution methods suitable for approximate solution of equations of this type on UW MIPAC's various minicomputers. The optimal control problem being studied concerns the production of relatively "pure" soliton solutions, minimally corrupted by the presence of other solution forms, by means of boundary inputs to the system.

Professor Dahlard L. Lukes, of the University of Virginia, visited UW MIPAC during Sem I, 1987 - 1988 as a Research Consultant, supported in part by funds provided by the subject grant. Professor Lukes carried out a program of research in nonlinear differential equations and nonlinear control systems while here. The Principal Investigator particularly wishes to recognize his valuable contributions toward the successful completion of Thomas Svobodny's PhD. thesis.

iii) <u>Research Carried out by Research Assistants and Dissertators</u>
<u>Supported in part by the Subject Grant; PhD Graduates</u>.

During the subject grant period three students graduated with the PhD degree on the completion of theses supervised by the Principal Investigator. These are:

Dr. Thomas Svobodny (currently at Virginia Tech)

Dr. Robert Acar (currently at the University of Oklahoma)

Dr. Khosro Shabtaie

All three of these graduates were supported in part as research
assistants under this grant and were also given additional support
in the form of domestic travel, scientific computation support,
and funds for thesis preparation. Both Svobodny and Acar also made
use of UW MIPAC experimental equipment to obtain data for their
thesis work.

Svobodny's thesis is concerned with state estimation, and to
some extent with system stabilization based on that estimation, for
a variety of nonlinear systems, particularly those which exhibit
limit cycles, or self-excited oscillations. Systems of this sort
are important in a wide variety of applications, including those
associated with the study of aircraft flutter problems, general
problems of fluid flow at high Reynolds number, etc. His work ex-
plores a number of state estimation schemes and establishes their
validity in the nonlinear oscillator context, with appropriate
assumptions, the most important of which is a dissipativity assump-
tion on the resulting system satisfied by the estimator error. A
number of examples are analyzed in detail, including the Van der Pol
equation and elastic systems coupled to disturbances generated by
a system of Van der Pol type. Actual data taken from an experimen-
tal airfoil was used in one section of the thesis.

Acar's dissertation is concerned with identification of the so-
called "permeability" or "transmissivity" coefficient p and the in-
homogeneous "source" term f in the elliptic equation

$$\text{div } (p \cdot \text{grad } \varphi) = f \ .$$

This problem is important in a wide variety of applied contexts, in-
cluding the study of groundwater flow, petroleum exploration and re-
covery enhancement, identification of elastic structures in space
applications, etc. Acar has developed in this thesis a weak formu-
lation of an equation error approach which has proved computation-
ally effective in many examples, a number of which were dealt with
in some detail in the thesis itself. Acar's thesis also treats cer-
tain cases where the function p may undergo jump discontinuities
along curves in the region of interest. Procedures to identify the
discontinuity curves were discussed, as well as modifications of the
identification procedure to allow estimation of the discontinuities
of p along such curves.

Shabtaie's work is concerned with identification of damping and

elasticity coefficients in a structurally damped beam modelled by a modified Euler-Bernoulli equation of the form

$$\rho(x)\frac{\partial^2 w}{\partial t^2} - 2\gamma \frac{\partial^3 w}{\partial t \partial x^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w}{\partial x^2} \right] = g(x)\ v(t)\ ,$$

where ρ is the linear mass density distribution, γ is the damping coefficient, $EI(x)$ is the bending modulus distribution, and $v(t)$ is an external excitation whose effect is introduced into the system by means of the function, or distribution, g. A new procedure, which we call the estimator - predictor technique, based on the effectiveness of the model when incorporated into a state estimator system, is analyzed in the thesis and used for computations. This method accepts an output $y(t)$, over an interval $0 \leq t \leq T$ and, using an estimator based on the assumed coefficients in the model, attempts to predict $y(t)$ over the next interval $T \leq t \leq 2T$ through a generated output prediction $\eta(t)$. The coefficients are adjusted so as to minimize the error in the L^2 norm. A modification of the procedure allows for continuous, on-line, updating. The method was tested on data obtained from a steel beam, excited by an electromagnetic shaker in the UW MIPAC laboratory with excellent results. These results were compared with a previously known, and closely related method, the so-called "model reference" procedure, with comparisons quite favorable to the new technique.

iv) Experimental Research Carried out in the UW MIPAC Laboratory.

The greater part of laboratory research carried out during the subject grant period by the Principal Investigator has been done in connection with damping mechanisms in elastic beams. A variety of considerations; partial experimental results and certain proposed mathematical models for the damping process, a resulting in a situation where our assumptions on the nature of the damping process must be examined very carefully. In particular, intensive experimental studies need to be undertaken to provide a basis for choice between a strictly frequency proportional damping, as we have been working with for some years now, and a more complex model, for which we suggest the term *modified Kelvin - Voigt damping*, wherein the rate of growth of damping relative to frequency is quadratic at low frequencies, approximately linear in the intermediate ranges, and asymptotically constant as the frequency tends to infinity. The

latter model appears, from experiments carried out earlier, to ag-
ree with what we observe in longitudinal vibrations of steel bars
and agrees with the predictions of thermoelastic damping models.

Our present experiments are being carried out with very careful
attention to the elimination of external energy losses or, where
this is not possible, to estimation of the magnitude of those losses.
We are measuring the vibrations of the elastic bodies themselves with
a non-contact laser vibrometer, to eliminate accelerometer attachment
losses known to have occurred in earlier work with accelerometers
mounted on the samples. We have concluded that it is not practical
to attempt to avoid losses through propagated sound waves by placing
the sample in a vacuum. This would necessitate attachment of an ac-
celerometer or else a radio device, as used by some earlier experi-
menters in vacuum experiments, in order to record the vibrations.
Even if the vacuum chamber were supplied with a glass window for ob-
servation, it would not be possible to take laser vibrometer measure-
ments through the glass because of double - reflection problems en-
countered in such situations. Instead, we are directly measuring
the intensity of sound waves generated in the air by beam vibrations
with the use of very accurate microphones placed near the sample.
This appears to be feasible because it is not necessary, in general,
to determine the absolute degree of energy loss to the air; it is
only necessary to determine its dependence on frequency in order to
answer the questions which appear to be the most important ones to
ask at the present time.

v) <u>Joint Experimental Research Carried out with UW MIPAC Visitors
Partially Supported by the Subject Grant</u>.

Joint laboratory experimentation has been carried out with two
visitors to UW MIPAC; Professor G. Chen, formerly with Pennsylvania
State University and now with Texas A. & M. University, and Professor
Luther White of the University of Oklahoma.

Professor Chen has been undertaking extensive studies of segmen-
ted beams, consisting of continuous sections of uniform length con-
nected at joints, which have different elastic and damping properties
as compared with the beam sections themselves. The periodic charac-
ter of these elastic structures is reflected in complementary period-
ic behavior of the natural frequencies and of the damping rates. In

connection with a recent mathematical paper which Professor Chen and
co-authors, including the Principal Investigator, have prepared on
this subject, a variety of experiments were carried out with the use
of UW MIPAC facilities. These served to confirm, in large part, the
predictions of the mathematical theories presented in the; the exper-
imental results are reported along with the theoretical material in
the article for publication.

vi) <u>Installation and Testing of New UW MIPAC Equipment</u>.

The grant period being reported on is noteworthy, as far as
UW MIPAC experimental facilities is concerned, because it marked
the beginning of installation of new equipment funded under AFOSR
participation in the 1986-87 DoD Universities Instrumentation Pro-
gram. The new facilities include:

i) An MTS Systems, Inc., hydraulically actuated vibration test
platform;

ii) A Dantec laser vibrometer system;

iii) A Hewlett Packard 3562 vibration analyzer;

iv) An IBM PC/AT microcomputer with AD/DA conversion facilities;

v) A Hewlett Packard 310 microcomputer with modal analysis
software.

Together, these new facilities provide UW MIPAC with a badly needed
non-contact vibration measurement capability and the beginnings of
an active vibration control capability. In fact, all of the pieces
of the vibration control facility are now in place except AD/DA con-
version for the HP 310 microcomputer, which serves as the control
and monitoring unit for the control system based on the MTS vibration
test platform.

All units have now been installed and are now in the initial
test phase of operation. It is expected that the laser vibrometer
system will be used to a considerable extent in the forthcoming
May 1988 UW MIPAC *Workshop on Experimental and Computational Aspects
of Control*.

vii) <u>Seminars, Invited Speakers</u>.

UW MIPAC has for some time sponsored a seminar on mathematical modelling and control, which meets Wednesday afternoons during the academic year. In addition to regular presentations by the Principal Investigator, associates and research assistants, a number of visiting speakers from other institutions are invited, from time to time, to give guest lectures. During the period covered by this report, guest lectures were given by

Ruth Curtain	Univ. of Groningen, Netherlands
Elena Fernandez	Virginia Tech
Gunter Leugering	Technische Hochschule Darmstadt
Dahlard Lukes	University of Virginia
Marshall Slemrod	Rensselaer Polytechnic Institute
Eduardo Sontag	Rutgers University
Robert Wheeler	Virginia Tech
Luther White	University of Oklahoma

These talks added greatly to the breadth and variety of the UW MIPAC program.

viii) <u>Documentation and Bibliographical Efforts</u>.

Over the years the Principal Investigator, like almost everyone else in this field, has accumulated a vast collection of reprints, preprints, reports, etc. These contain a large quantity of potentially valuable information, provided it can be identified and retrieved. A relatively modest sum from grant funds has been used to hire Miss Barbara Tavis to organize the scientific documents collection so as to make it accessible and usable. An efficient, computerized system has been devised which enables location of manuscripts quickly and efficiently by means of key words supplied to any one of MIPAC's computers on campus.

ix) <u>Scientific Travel, Lectures</u>.

The Principal Investigator, supported in part by g.

travelled to Paris France, in June, July, 1988, to attend the First
International Conference on Industrial and Applied Mathematics. A
paper on coefficient identification from transfer function data
was presented by invitation of the conference organizers. Later
travel on this same trip allowed presentation of two lectures at
Karl - Franzens Universitat in Graz, Austria.

A paper on damping in elastic systems was presented at the 1987
Control and Decision Conference in Los Angeles. Grant funds were
used to support this trip.

In March, 1987, a trip was made to the State University of New
York, Buffalo, in order to confer with Professor Dan Inman and ins-
pect his control laboratory there. Valuable insights potentially
useable by UW MIPAC were obtained from this trip.

Other travel included trips to Virginia Tech., Blacksburg, VA,
Carnegie Mellon, Pittsburgh, PA, and the University of Nebraska, in
Lincoln, NE.

APPENDIX C

"Some Remarks on Transfer Function Methods for Infinite

Dimensional Linear Systems"

by

David L. Russell

Principal Investigator

SOME REMARKS ON TRANSFER FUNCTION METHODS FOR INFINITE

DIMENSIONAL LINEAR SYSTEMS*

by D. L. Russell**

ABSTRACT

This paper develops certain properties of transfer function representations of a class of distributed parameter systems generally lying outside the well known *Callier - Desoer* class. We are particularly concerned with the question of representation of closed loop semigroups in terms of the resolvent of the original open loop generator and the closed loop transfer function, frequency domain criteria for input and output admissibility, etc. It develops that transfer functions of the distributed parameter systems considered here, and their associated input - output operators, may have undesirable properties with respect to approximations of various types when insufficient dissipation is present in the system; adequate dissipation mechanisms are studied in this connection.

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0.Introduction. The 1980's have seen a resurgence of interes in the transfer function description of linear input - output systems. These methods have a number of attractive properties, particularly in regard to questions of robustness with respect to system variations, as developed, e.g., in the papers of Doyle and Stein ([6]) and many others, too numerous to cite in any representative way.

The application of these methods, sometimes called H^{∞} methods because of the boundedness property in a right half plane characteristic of transfer functions of well-posed, nonanticipative linear systems, has been extended to infinite dimensional ("distributed parameter") systems only relatively recently. We cite the works of Callier and Desoer ([2]), Curtain and Pritchard ([5]), Pritchard and Salamon ([15]),and Francis ([10]). Extensive references are presented in Curtain's expository article, ([4]), as well as in Francis' book, already cited.

The majority of "H^{∞}" studies for infinite dimensional systems have been conducted in the context of systems possessing transfer functions in the so-called *Callier - Desoer* class ([2],[3], and [4]), or to what Curtain identifies as the *Pritchard - Salamon* class ([4]). These classes both exclude transfer functions arising from a number of systems important in applications, such as undamped, or lightly damped, elastic systems described by partial differential equations of hyperbolic type, neutral functional equations, etc.

The present paper does not treat H^{∞} control synthesis methods; rather, it aims to develop certain properties of transfer functions of some systems which do not belong to the Callier - Desoer class. It turns out that transfer functions of systems such as those considered here may have some rather unusual, one might even say disconcerting, properties, particularly in relation to convergence questions with respect to approximation via finite dimensional systems. In addition to these approximation questions, we will discuss the use of transfer function methods to analyze closed - loop systems arising out of certain linear feedback laws, use of transfer functions to determine the admissibility of input and output mechanisms, and a number of other, related, matters. In the process we develop a representation of the semigroup in terms of its inverse Laplace transform which, in its specific context, is less restrictive than the general treatment of that matter by Dunford and Schwartz ([8]). The primary motivation

for the development of this article may be summarized in the following way. The recent success of frequency domain methods virtually guarantees that a large fraction of future distributed parameter control research will be carried out within this framework. That granted, it seems important to us that the connections between frequency domain representations and their state space counterparts should be fully explored, with appropriate cautions posted where those connections appear to be somewhat tenuous. We are hopeful that the work presented here will represent a useful step in that direction. A logical direction for future work is to develop H^{∞} synthesis techniques for systems of the sort described here, including co-prime factorization techniques, etc., taking into account the properties which we develop here.

The present paper is, for convenience, written in the single input – single output context but most results obtained here extend rather directly to systems with a finite number of inputs and outputs. This is certainly not the case for H^{∞} methods in general because of the non-commutativity of matrix multiplication, but that noncommutativity plays a minor role relative to the topics of interest to us here.

1. Transfer Functions and Admissibility of Input and Output Elements

Let us consider a controlled and observed linear system

$$\dot{x} = A x + b u , \tag{1.01}$$

$$y = \langle x , c \rangle \equiv c^* x , \tag{1.02}$$

with state x in a Hilbert space X (norm $\| \ \|$, inner product $(\ , \ )$), u and y scalar (this is for convenience only; most of our results extend immediately to u, y $\in R^m$). We suppose that the closed operator A is the generator of a strongly continuous semigroup, S(t), of bounded operators on X.

We assume there is a second Hilbert space, W, continuously and densely embedded in X, which includes the domains of both A and A^*. We take W' to be the dual of W relative to X (see [16], e.g.) and thus have the familiar inclusions

$$W \subset X \subset W' . \tag{1.03}$$

Each element of W is a continuous linear functional on W; relative to the norm of X each such element will, in general, be an unbounded linear functional. We will assume that W includes the domains of both the generator, A, and its adjoint, A^*.

We assume that b is an *admissible input element* for the system (1.01), to be understood as developed in [11]. Thus we take $b \in W'$; in particular then, b is defined on the domain of A*. The value $\langle x,b \rangle$ of the linear functional b at x , which we will also designate as $b^* x$ for notational convenience, is defined when x is any element of W, in particular any element in the domain, $\mathcal{D}(A^*)$, of A^*. The primary requirement for the admissibility of b as an input element for (1.01) is that for each T > 0 the map $B_T : L^2[0,T] \to X$ defined in a dualistic manner by

$$\eta^* B_T \, u \;=\; \int_0^T (b^* S(T-t)^* \eta)^* u(t) dt, \quad \eta \in D(A^*), \qquad (1.04)$$

should correspond to a bounded operator. Sufficient conditions for this - the Carleson measure criteria - were developed in [11].

If b is an admissible input element for (1.01), that system has a unique "generalized" or "mild" solution x(t), continuous as a function from R^1 to X, corresponding to each initial state $x(0) = x_0 \in X$ and each input function $u \in L^2[0,T]$ for each $T > 0$ (i.e. locally square integrable). That solution is given by the formula

$$x(t) = S(t) \, x_0 \;+\; \int_0^t S(t-s) \, b \, u(s) \, ds \,, \qquad (1.05)$$

where the interpretation of the integral is just

$$\int_0^t S(t-s) \, b \, u(s) \, ds \;\equiv\; B_t \, u$$

as discussed above.

An *admissible output element* (unfortunately that was not defined in [11]) c^* is also defined in a duality framework; such an element c is an admissible input element, as defined above, for the dual system

$$\dot{y} \;=\; A^* y \;+\; c \, v \,.$$

Thus c also lies in W' and thus, in particular, is defined on the domain of A. The operator C_T defined by

$$C_T \, v \;=\; \int_0^T S(T-s)^* \, c \, v(s) \, ds$$

is then bounded for each $T > 0$. Of more direct interest in the observation, or output, context is the fact that the dual operator,

C_T^*, defined pointwise for $x \in \mathcal{D}(A)$ by

$$y(t) = (C_T^* x)(t) = c^* S(t) x , \qquad (1.06)$$

extends to a bounded operator from X to $L^2[0,T]$.

Defining the Laplace transforms of $u(t)$ and $y(t)$ in the usual way for complex λ, and denoting the transforms by $\hat{}$, we have the relation

$$\hat{y}(\lambda) = T(\lambda) \hat{u}(\lambda) , \qquad (1.07)$$

where, with appropriate extension of the domain of the resolvent operator $R(\lambda) = (\lambda I - A)^{-1}$, as will be discussed more fully in a later section,

$$T(\lambda) = c^* R(\lambda) b . \qquad (1.08)$$

From standard Laplace transform theory (e.g.,[20]) it follows that the convolution "input-output" operator

$$(Ju)(t) = \int_0^t c^* S(t-s)b\, u(s)\, ds, \qquad (1.09)$$

where $c^* S(t)b$, which may be a bounded measure rather than a function, is defined as the inverse Laplace transform of $T(\lambda)$. It is familiar that (1.09) defines a bounded operator

$$J_\rho: L^2[0,\infty) \rightarrow L_\rho^2[0,\infty), \qquad (1.10)$$

where for each real ρ

$$L_\rho^2[0,\infty) = \left\{ f \in L_{\ell oc}^2[0,\infty) \mid \int_0^\infty e^{-2\rho t} \|f(t)\|^2\, dt < \infty \right\} , \qquad (1.11)$$

just in case $T(\lambda)$ acts as a bounded multiplication operator from the

Hardy space $H^2(\mathbb{C}^+)$ of the right half plane to $H^2(\mathbb{C}^+_\cup)$, the corres-
ponding space in the half plane Re $\lambda \geq \rho$. This, in turn, is true
if and only if $T(\lambda)$ is a uniformly bounded analytic function in
that half plane. When these conditions obtain we will say that b
and c form a *jointly admissible* pair of input and output elements.
Another possible definition of joint admissibility of b and c rel-
ative to the system (1.01),(1.02) is that (1.09) defines a bounded
operator

$$J_T : L^2[0,t] \to L^2[0,T] \qquad\qquad (1.12)$$

for each $T > 0$. We will see below that, as one might anticipate,
this is equivalent to the definition already given. The input and
output elements b and c, rexpectively, may be jointly admissible
without being individually admissible as input and output elements.
We will say that b and c are *totally admissible* for the system
(1.01),(1.02) if they are jointly and individually admissible as
described above.

A very important special case occurs when the operator A is a
discrete spectral operator with a complete set of eigenvectors,

$$\Phi = \{ \varphi_k \mid k = 1,2,3, \ldots \} ,$$

with corresponding eigenvalues λ_k of single multiplicity, forming a
Riesz basis ([25]) in X. This means that each $x \in X$ has a unique
X - convergent representation

$$x = \sum_{k=1}^{\infty} \xi_k \, \varphi_k , \qquad\qquad (1.13)$$

and there are positive numbers m,M, independent of x, such that

$$m^2 \, \|x\|^2 \leq \sum_{k=1}^{\infty} |\xi_k|^2 \leq M^2 \, \|x\|^2 .$$

One may proceed then to show quite easily (see, e.g., [17]) that
there exists a unique dual Riesz basis in X;

$$\Psi = \{ \psi_\ell \mid \ell = 1,2,3, \ldots \} ,$$

biorthogonal to Φ, i.e.,

$$(\varphi_k , \dot{\psi}_\ell) = \delta_{k\ell} ,$$

consisting of eigenvectors of the adjoint operator A^* corresponding to its eigenvalues λ_k. Assuming that $\lim\limits_{k\to\infty} |\lambda_k| = \infty$, the domain of A is

$$\mathfrak{D}(A) = \left\{ x = \sum_{k=1}^{\infty} \xi_k \varphi_k \ \Big| \ \sum_{k=1}^{\infty} |\lambda_k \xi_k|^2 < \infty \right\}$$

and, with this domain, A is clearly closed. The Hille – Yoshida condition ([8]) implies there is a real number a such that

$$\text{Re } \lambda_k \leqq a, \quad k = 1,2,3, \ldots . \tag{1.14}$$

Since $\mathfrak{D}(A^*)$ is assumed to lie in W we may define

$$\overline{b}_k = b^* \dot{\psi}_k , \quad k = 1,2,3, \ldots ,$$

and we may identify (made more specific in [11]) b with the series

$$b = \sum_{k=1}^{\infty} b_k \varphi_k .$$

We will refer to the b_k as the *input coefficients* associated with b and the system (1.01). If the state x(t) for (1.01), (1.05) is represented as in (1.13) it may be seen that its k-th coefficient is

$$\xi_k(t) = e^{\lambda_k t} \xi_{k,0} + b_k \int_0^t e^{\lambda_k(t-s)} u(s) \, ds ,$$

where the inital state x_0 has the representation, similar to (1.11),

$$x_0 = \sum_{k=1}^{\infty} \xi_{k,0} \varphi_k .$$

Consider now a solution $x(t)$ of the homogeneous counterpart

$$\dot{x} = A x \qquad (1.15)$$

of (1.01); the output via (1.06) is then just

$$y(t) = \sum_{k=1}^{\infty} \bar{c}_k e^{\lambda_k t} \xi_{k,0} ,$$

where the *output coefficients* c_k , $k = 1,2,3, \ldots$, are defined by

$$\bar{c}_k = c^* \varphi_k .$$

For the linear observed system (1.02), (1.03) with initial state $x(0) = 0$ we have the output (cf. (1.06))

$$y(t) = \sum_{k=1}^{\infty} \bar{c}_k \int_0^t e^{\lambda_k(t-s)} b_k u(s) ds , \qquad (1.16)$$

at least formally. The transfer function, $T(\lambda)$, is the Laplace transform of the *impulse response function*

$$\sum_{k=1}^{\infty} \bar{c}_k b_k e^{\lambda_k t}$$

and may be identified, under certain restrictions, with the series

$$\sum_{k=1}^{\infty} \frac{\bar{c}_k b_k}{\lambda - \lambda_k} , \qquad (1.17)$$

concerning which we will have more to say later.

We have remarked that admissibility of the input and output elements b and c, by itself, fails to guarantee any sort of joint admissibility. For instance, let us consider a system (1.01), (1.02) having an orthonormal basis of eigenvectors φ_k, $k = 1,2,3, \ldots$, and eigenvalues

$$\lambda_k = -k, \quad k = 1,2,3, \ldots .$$

An example of this type is given in [11]. We suppose an input element b is used for which the coefficients, as described above, are

$$b_k = (-1)^{k-1}, \quad k = 1,2,3, \ldots .$$

The Carleson criterion developed in [11] shows that such an input element is admissible. By the same criterion, if we take c = b, so that (1.02) becomes

$$y = c^* x = b^* x ,$$

the coefficients c_k are the same as the b_k and, at least formally,

$$T(\lambda) = \sum_{k=1}^{\infty} \frac{\overline{c}_k \, b_k}{\lambda + k} = \sum_{k=1}^{\infty} \frac{1}{\lambda + k} .$$

As this series is everywhere divergent, b and c = b do not form a jointly admissible pair of input and output elements in this case.

For the same input element, b, if we select c so that

$$c_k = 1, \quad k = 1,2,3, \ldots ,$$

then

$$T(\lambda) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\lambda + k} .$$

Since

$$\frac{1}{\lambda+k-1} - \frac{2}{\lambda+k} + \frac{1}{\lambda+k+1} = \frac{2}{(\lambda+k)((\lambda+k)^2-1)} ,$$

we may, by grouping, write $T(\lambda)$ as

$$T(\lambda) = \frac{1}{2} \left[\frac{1}{\lambda+1} + \sum_{\ell=1}^{\infty} \left(\frac{1}{\lambda+2\ell-1} - \frac{2}{\lambda+2\ell} + \frac{1}{\lambda+2\ell+1} \right) \right]$$

$$= \frac{1}{2(\lambda+1)} + \sum_{\ell=1}^{\infty} \frac{1}{(\lambda+2\ell)((\lambda+2\ell)^2-1)} \, ,$$

which converges uniformly in compact subsets of Re $\lambda \geq 0$ and is uniformly bounded by the value attained with $\lambda = 0$, i.e.,

$$\frac{1}{2} \left[1 + \sum_{\ell=1}^{\infty} \frac{1}{\ell(4\ell^2-1)} \right] \, .$$

Thus, unlike the situation which obtains for admissible input and output elements, it would appear to be not feasible to find a general growth condition on the $|b_k|$ and $|c_k|$ stringent enough to guarantee joint admissibility and yet wide enough in scope to admit most of the systems studied in applications. Any adequate criterion will have to take the signs of the coefficients into account and allow for conditional convergence of the series representation of the transfer function $T(\lambda)$.

In the discussion above we have treated input output operators in two different contexts; we have considered J_T , as in (1.12), in our "time domain" discussion and we have considered operators J_a defined in a slightly different way in (1.10), related to the "frequency domain" formulation expressed in terms of $T(\lambda)$. The following theorem explores the equivalence of these operators.

Theorem 1.1 *Let b and c be totally admissible and let e^{At} satisfy the growth condition*

$$\|e^{At}\| \leq M \, e^{at} \, , \quad t \geq 0,$$

for some real a. Then the operator J_T defined by (1.12) is bounded for each T > 0 just in case for every real β and every α $\geq$ a the operator

$$J_{\beta,\rho}: \ L_{\beta}^2[0,\infty) \to L_{\rho}^2[0,\infty), \quad \rho = \max \{\alpha,\beta\}, \qquad (1.18)$$

also defined in terms of (1.09), *is bounded.*

<u>Proof</u>. Inferring the boundedness of J_T from that of $J_{\beta,\rho}$ is trivial, as we see by restricting the support of u to [0,T], observing y only on [0,T] also and noting that the space $L_r^2[0,T]$ is equivalent to the space $L^2[0,T]$, for any real r, if the support of the functions considered is restricted to the finite interval [0,T] in question.

To show that the boundedness of $J_{\beta,\rho}$, as defined, follows from that of J_T, T > 0, we choose some T > 0 and then define n(t), t > 0, to be the largest integer such that n(t)T < t. Then, with the integrals below interpreted in the way stipulated earlier, we have

$$x(t) = \int_0^t e^{A(t-s)} \ b \ u(s) \ ds \ = \ \sum_{n=0}^{n(t)-1} \int_{nT}^{(n+1)T} e^{A(t-s)} \ b \ u(s) \ ds$$

$$+ \int_{n(t)T}^t e^{A(t-s)} \ b \ u(s) \ ds$$

$$= \ (\text{setting } t = \tau + (n+1)T, \ s = \sigma + (n+1)T)$$

$$= \ \sum_{n=0}^{n(t)-1} e^{A(t-(n+1)T)} \int_0^T e^{A(T-\sigma)} \ b \ u_n(\sigma) \ d\sigma$$

$$+ \int_{n(t)T}^t e^{A(t-s)} \ b \ u(s) \ ds \ ,$$

where $u_n(\sigma) = u(\sigma+(n+1)T)$ for each indicated n. Correspondingly,

$$y(t) \ = \ \sum_{n=0}^{n(t)-1} c* \ e^{A(t-(n+1)T)} \int_0^T e^{A(T-\sigma)} \ b \ u_n(\sigma \ d\sigma$$

$$+ c^* \int_{n(t)T}^{t} e^{A(t-s)} \, b \, u(s) \, ds \; . \qquad (1.19)$$

If we assume the boundedness of J_T and the admissibility of b and c, then for $u \in L_\beta^2[0,\infty)$ the integrals in (1.19) represent points in the Hilbert space X whose norms can be bounded in terms of the norms of the u_n in $L^2[0,T]$, hence by a multiple of $e^{\beta nT}$. On the other hand the norm of the operator $c^* e^{A(t-(n+1)T)} : X \to L^2[mT,(m+1)T]$, where $m+1 = n(t)$ for each $t \in (mT,(m+1)T]$, can be bounded in terms of $e^{\alpha(m-n)T}$, $\alpha \geq a$. Combining the two estimates with $\rho = \max \{\alpha, \beta\}$, we can bound the output y in $L_\rho^2[0,\infty)$ in terms of the norm of u in $L_\beta^2[0,\infty)$ and the proof is complete.

2. Laplace Transform Representation of Solutions.

For the homogeneous system with given initial state,

$$\dot{x} = A x , \quad x(0) = x_0 \in X , \tag{2.01}$$

we have the familiar representation, for $t \geq 0$,

$$x(t) = S(t) x_0 ,$$

where $S(t)$ is the strongly continuous semigroup of bounded operators on X generated by A. If we form the Laplace transform

$$R(\lambda) = \int_0^\infty e^{-\lambda t} S(t) dt$$

for Re $\lambda > a$, as defined earlier, it is also familiar, and easily seen, that $R(\lambda)$ is the resolvent of the operator A,

$$R(\lambda) = (\lambda I - A)^{-1} .$$

Correspondingly, the Laplace transform of the solution $x(t)$ determined by (2.01) is

$$\hat{x}(\lambda) = R(\lambda) x_0 .$$

If we formally apply the Laplace inversion formula, integrating over the contour Γ_α consisting of the line Re $\lambda = \alpha$, oriented in the upward direction, we have, for $\alpha > a$,

$$x(t) = \frac{1}{2\pi i} \int_{\Gamma_\alpha} e^{\lambda t} R(\lambda) x_0 d\lambda, \tag{2.02}$$

corresponding to

$$S(t) = \frac{1}{2\pi i} \int_{\Gamma_\alpha} e^{\lambda t} R(\lambda) \, d\lambda. \tag{2.03}$$

However, as Kato points out in [12], it is not generally an easy matter to establish convergence of this integral in an appropriate and usable sense. Indeed, we need to insist on actual convergence of the integral (2.02) with respect to the norm of X, rather than in some analog of the usual l.i.m. sense, in order to be able to conclude, on the basis of this representation, that x(t) is a continuous vector function of t. Consequently we make the assumption, which we will verify at the end of this section for some important cases, that the integral (2.02) is the limit, with respect to the norm of X, of corresponding integrals over $\Gamma_{\alpha,R}$, the restriction of Γ_α to $|\text{Im } \lambda| \leqq R$, as R ranges over some sequence of values tending to infinity, and that the integral (2.03) converges, through the same truncation process, with respect to the strong operator topology of bounded linear operators on X (the two are, of course, the same notion).

Since an admissible input element b is, in particular, a linear functional on W, which includes the domain of A^*,

$$b^* R(\lambda)^* = b^* (\overline{\lambda}I - A^*)^{-1}$$

is a continuous linear functional on X for Re $\lambda > a$. Consequently $R(\lambda)$ b represents, for each such λ, an element of X. Taking λ_0 with real part $> a$ and using the resolvent identity we have

$$R(\lambda)b - R(\lambda_0)b = (\lambda_0 - \lambda)R(\lambda)R(\lambda_0)b \, .$$

Using the necessity of the Hille-Yoshida condition ([8]) we can bound the first two factors uniformly and we conclude that $R(\lambda)b$ is uniformly bounded in each closed half plane C_α^+, $\alpha > a$. Similar conclusions can be drawn in regard to $c*R(\lambda)$ and $R(\lambda)*c$ if c is an admissible output element. If $u \in L^2[0,\infty)$ and $\hat{u} \in H^2(C^+)$ is its Laplace transform, then the Laplace transform of the solution $x_u(t)$ of

$$\dot{x} = A x + b u, \quad x(0) = 0,$$

is easily seen to have the Laplace transform $R(\lambda) b \hat{u}(\lambda)$. In the other direction it can be seen that a sufficient condition for admissibility of the input element b is the boundedness of $R(\lambda)b$ together with convergence, in the sense described above, and uniformly in compact intervals of $t \geq 0$, of the inversion integral

$$x_u(t) = \frac{1}{2\pi i} \int_{\Gamma_\alpha} e^{\lambda t} R(\lambda) b \hat{u}(\lambda) d\lambda \qquad (2.04)$$

with respect to the topology of X for each $\hat{u} \in H^2(\mathbb{C}^+)$. It is not clear, in general, to what extent this is a necessary condition. However, since G. Weiss has established in [22] the necessity of the Carleson criterion (cf. [11]) when the operator A has a Riesz basis of eigenvectors corresponding to eigenvalues lying in a vertical strip in the complex plane, it seems likely that the related condition (2.04) will be found necessary under comparable conditions. We will see in an example later in this section that just the boundedness of $R(\lambda) b$ is not enough to guarantee that b is admissible. The corresponding output via an admissible output element c is

$$y_u(t) = \frac{1}{2\pi i} \int_{\Gamma_\alpha} e^{\lambda t} c^* R(\lambda) b \hat{u}(\lambda) d\lambda = \frac{1}{2\pi i} \int_{\Gamma_\alpha} e^{\lambda t} T(\lambda) \hat{u}(\lambda) d\lambda .$$

For joint admissibility of b and c we require only that the output y_u be in some $L_\alpha^2[0,\infty)$, α fixed, for all u in $L^2[0,\infty)$. This will be true if the integral converges in the l.i.m. sense and this, in turn, will be the case if the transfer function $T(\lambda)$ is uniformly bounded for Re $\lambda \geq \alpha$, in agreement with our earlier discussion in §1. Parallelling the counterexample of §1, the boundedness of $T(\lambda) = c*R(\lambda)b$ cannot, in general, be inferred from the boundedness of both $c*R(\lambda)$ and $R(\lambda)b$.

Very much of what we do in §3 depends on the convergence, in the space X, of the integrals (2.02) and (2.03). As we have already observed, it is not always easy to guarantee this. Dunford and Schwartz avoid this problem in [8] by replacing the integral (2.02) by

$$x(t) = \frac{1}{2\pi i} \int_{\Gamma_\alpha} e^{\lambda t} \frac{R(\lambda)(\mu I - A)^2 \, d\lambda}{(\mu - \lambda)^2} \, x_0 \, ,$$

valid for $x_0 \in \mathfrak{D}(A^2)$. The last restriction poses a problem for us in this paper because the feedback operations we wish to study in general change the domain of A. There may be some way to handle this but we do not pursue that route here.

We examine first some special, but reasonably extensive, cases wherein our convergence assumptions are satisfied for (2.02), (2.03). We first consider the case wherein S(t) is a holomorphic semigroup on X. Here the spectrum of A lies in a wedge shaped region

$$\Sigma_{\alpha, \psi + \varepsilon} = \left\{ \lambda \mid \lambda = a + r \, e^{i\theta} \, , \, \psi + \varepsilon \leq |\theta| \leq \pi \, , \, 0 \leq r < \infty \right\} \quad (2.05)$$

where $\pi/2 < \psi < \pi$ and ε is a conveniently small positive number. Then, as is shown in [12], for $t > 0$,

$$S(t) = \frac{1}{2\pi i} \int_{\Gamma_{\alpha, \psi}} e^{\lambda t} \, R(\lambda) \, d\lambda \quad (2.06)$$

where

$$\Gamma_{\alpha, \psi} = \left\{ \lambda \mid \lambda = \alpha + r \, e^{\pm i\psi} \, , \, 0 \leq r < \infty \right\} \, ,$$

oriented in the direction of increasing imaginary part. Since it is a familiar fact in the holomorphic case that, for some $M_\psi > 0$,

$$\|R(\lambda)\| \leq M_\psi \, / \, |\lambda| \, , \, \lambda \in \overline{\mathbb{C} - \Sigma_{\alpha, \psi}} \, ,$$

it is clear that the integral converges, in fact, with respect to the uniform operator topology. Cauchy's theorem, plus some fairly easy estimates, then enable us to see that (2.06) remains true with the path $\Gamma_{\alpha, \psi}$ replaced by Γ_α described earlier, the converg-

ence remaining true in the uniform operator topology.

An example from this category shows that the boundedness of $R(\lambda)$ b is not enough to yield admissibility of the input element b. The example involves the one-dimensional heat equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = 0 \, , \, t > 0, \, 0 < x < \pi$$

with boundary conditions

$$w(0,t) = 0, \, w(\pi,t) = u(t).$$

The state space is $X = L^2[0,\pi]$. A detailed discussion appearing in [11] identifies the input element b with the distribution δ_π' in this case and demonstrates that b is not an admissible input element. However, since the domain of

$$A = A^* = - \frac{\partial^2}{\partial x^2}$$

is a closed subspace of the Sobolev space $H^2[0,\pi]$ consisting of w which satisfy the corresponding homogeneous boundary conditions, we can take $W = H^2[0,\pi]$ and then $b \in W'$. Hence, as shown earlier, $R(\lambda)$ b is bounded - in this case for Re $\lambda \geq \alpha > -1/4$. Since the convergence of (2.04) in X is sufficient for admissibility of b it follows that this convergence cannot take place in this case for all $\hat{u} \in H^2(\mathbb{C}^+)$.

Thus to proceed further, even in the analytic case, we have to introduce additional structure into our framework. It seems likely that the restrictions thereby incurred can be relaxed, particularly in the direction of making allowance for continuous spectra.

Accordingly, we now suppose that the operator A has a *Riesz basis* ([24]) of eigenvectors

$$\Phi = \left\{ \varphi_k \; \middle| \; -\infty < k < \infty \right\}$$

corresponding to distinct eigenvalues λ_k, indexed in the same way and arranged so that the imaginary parts are nondecreasing as k increases (modifications are easily made to allow for finite multiplicity; we leave this to the interested reader). We further suppose that there are real numbers a, b, a > b, such that for all k as indicated,

$$b \leq \text{Re } \lambda_k \leq a$$

and that there are positive numbers D_0 and D_1 such that the number, $N(y_1,y_2)$, of eigenvalues λ_k with $y_1 \leq \text{Im } \lambda_k \leq y_2$ satisfies

$$N(y_1,y_2) \leq D_0 + D_1(y_2 - y_1) \tag{2.07}$$

whatever real values may be assumed by y_1 and $y_2 > y_1$.

As is well known ([17]) there is a unique dual Riesz basis

$$\Psi = \left\{ \psi_k \mid -\infty < k < \infty \right\},$$

consisting of eigenvectors of A^*, admitting a biorthogonal relationship

$$\psi_\ell^* \varphi_k \ \left[\ = (\varphi_k, \psi_\ell) \right] \ = \ \delta_{k\ell} \ , \ -\infty < k,\ell < \infty \ .$$

The operator A has the representation

$$A = \sum_{k=-\infty}^{\infty} \lambda_k \ \varphi_k \psi_k^* \tag{2.08}$$

and the resolvent and identity operators can be written

$$R(\lambda) = \sum_{k=-\infty}^{\infty} \frac{1}{\lambda - \lambda_k} \varphi_k \psi_k^* \ , \quad I = \sum_{k=-\infty}^{\infty} \varphi_k \psi_k^* \ ,$$

the first converging in the uniform operator topology, the second in the strong operator topology.

Let $\Gamma_{\alpha,\beta}$ be the contour in the complex plane consisting of the lines

$$\Gamma_{\alpha} = \{\ \lambda\ |\ Re\ \lambda = \alpha > a\ \}, \quad \Gamma_{\beta} = \{\ \lambda\ |\ Re\ \lambda = \beta < b\ \}\ ,$$

the first oriented in the upward direction, the second downward. The restrictions on the distribution of the eigenvalues λ_k are easily seen to imply the existence of a sequence $\{\ y_n\ |\ -\infty < n < \infty\ \}$ with

$$\lim_{n\to-\infty} y_n = -\infty\ , \quad \lim_{n\to\infty} y_n = \infty\ ,$$

and such that, for some positive number G,

$$|\ Im\ \lambda_k - y_n\ | \geqq G,\ -\infty < k < \infty\ ,\ -\infty < n < \infty\ . \tag{2.09}$$

For $m < n$, we define $\Gamma_{\alpha,\beta}^{m;n}$ to be the rectangular contour consisting of the intersection of $\Gamma_{\alpha,\beta}$ with $y_m \leq Im\ \lambda \leq y_n$ along with the obvious segments of $Im\ \lambda = y_m\ (y_n)$, the whole oriented positively. The residue calculus shows that the operator valued integrals

$$P_{m,n} = \frac{1}{2\pi i} \int_{\Gamma_{\alpha,\beta}^{m,n}} R(\lambda)\ d\lambda$$

yield projections on A - invariant subspaces $X_{m,n} \subset X$ and $P_{m,n}$ converges strongly to the identity operator, I, as $m \to -\infty$ and $n \to \infty$.

In order to study convergence as $m \to -\infty$, $n \to \infty$, we suppose that

$$x = \sum_{k=-\infty}^{\infty} x_k\ \varphi_k\ \in\ X\ .$$

Then

$$P_{m,n}\ x = \frac{1}{2\pi i} \int_{\Gamma_{\alpha,\beta}^{m,n}} R(\lambda)\ x\ d\lambda\ .$$

On Im $\lambda = y_n$ we can see that

$$\|R(\lambda)x\|^2 = \left\| \sum_{k=-\infty}^{\infty} \frac{x_k}{\lambda - \lambda_k} \varphi_k \right\|^2 = \sum_{k=-\infty}^{\infty} \frac{|x_k|^2}{(\mu-\mu_k)^2 + (y_n-\nu)^2} \, ,$$

where $\lambda = \mu + i\, y_n$, $\lambda_k = \mu_k + i\, \nu_k$. We can use the Riesz proper of the φ_k to see that

$$\|R(\lambda)x\|^2 \leq \sum_{k=-\infty}^{\infty} \frac{|x_k|^2}{(y_n - \nu_k)^2} \leq \|x\|^2 \sum_{k=-\infty}^{\infty} \frac{1}{(y_n - \nu_k)^2} \quad (2.10)$$

and the sum on the right is readily seen to be bounded, uniformly with respect to n, by using density bound (2.07) on the distribution of the λ_k together with (2.09). But we can obtain a more refined es-estimate than this. We let $\varepsilon > 0$ and let y_n and η_n be chosen so that

$$|x_k|^2 < \varepsilon \ , \quad k \geq \frac{|\eta_n|}{2} \ , \quad\quad (2.11)$$

$$\frac{1}{(y_n - \nu_k)^2} < \varepsilon \ , \quad |\eta_n - k| \geq \frac{|\eta_n|}{2} \ . \quad\quad (2.12)$$

Since for all k

$$|\eta_n| \leq |\eta_n - k| + |k| \ ,$$

the two sets of indices described by (2.11) and (2.12) cover all values of k, with some possible overlap, and we have

$$\sum_{k=-\infty}^{\infty} \frac{|x_k|^2}{(y_n-\nu_k)^2} \leq \sum_{k \geq \frac{|\eta_n|}{2}} \frac{|x_k|^2}{(y_n - \nu_k)^2} + \varepsilon \sum_{|\eta_n-k| \geq \frac{\eta_n}{2}} \frac{|x_k|^2}{(y_n - \nu_k)^2}$$

$$\leq \varepsilon \sum_{k=-\infty}^{\infty} \frac{1}{(y_n - \nu_k)^2} + \varepsilon \sum_{k=-\infty}^{\infty} |x_k|^2 \leq N \, \varepsilon \ .$$

for some $N > 0$, which may depend upon the vector x. Since we can make ε as small as we please by taking y_n sufficiently large, we see that

$$\lim_{\substack{n\to\infty \\ \text{Im } \lambda = y_n}} \sup \|R(\lambda)x\| = 0 \qquad (2.13)$$

and we obtain a similar result on Im $\lambda = y_m$ as $m \to -\infty$. Letting $n \to \infty$ and $m \to -\infty$, (2.13) is seen to imply that for all $x \in X$

$$x = \frac{1}{2\pi i} \int_{\Gamma_{\alpha,\beta}} R(\lambda) \, x \, d\lambda \ .$$

Thus

$$I = \frac{1}{2\pi i} \int_{\Gamma_{\alpha,\beta}} R(\lambda) \, d\lambda \ ,$$

the integral converging with respect to the strong operator topology in the sense implied above.

If $x \in X$ and $f(\lambda)$ is a bounded holomorphic function on the union of $\Gamma_{\alpha,\beta}$ and its interior one may in the same way define

$$f(A) \, x = \frac{1}{2\pi i} \int_{\Gamma_{\alpha,\beta}} f(\lambda) \, R(\lambda) \, x \, d\lambda$$

so that

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma_{\alpha,\beta}} f(\lambda) \, R(\lambda) \, d\lambda \ .$$

In particular, for any initial vector $x_0 \in X$ we have, for $t \geq 0$,

$$S(t) \, x_0 \; = \; e^{At} \, x_0 \; = \; \frac{1}{2\pi i} \int_{\Gamma_{\alpha,\beta}} e^{\lambda t} \, R(\lambda) \, x_0 \; d\lambda \; .$$

$$= \; \lim_{\substack{m \to -\infty \\ n \to \infty}} \frac{1}{2\pi i} \int_{\Gamma_{\alpha,\beta}^{m,n}} e^{\lambda t} \, R(\lambda) \, x_0 \; d\lambda \; ,$$

again with respect to the strong operator topology.

Fixing m and n, we observe from our earlier argument that the bound (2.10) on $\|R(\lambda)x\|$ is independent of β. So, using the decay property of the exponential function valid for $t > 0$,

$$\lim_{\beta \to -\infty} \int_{\Gamma_{\alpha,\beta}^{n}} e^{\lambda t} \, R(\lambda) \, x_0 \; d\lambda \; \equiv \; \int_{\Gamma_{\alpha}^{n}} e^{\lambda t} \, R(\lambda) \, x_0 \; d\lambda \; ,$$

where the single index n indicates the line segment forming the upper boundary of the rectangle $\Gamma_{\alpha,\beta}^{m,n}$ in the first instance and its extension toward Re $\lambda = -\infty$ in the second. Then, using the estimate (2.13), we have

$$\lim_{n \to \infty} \left\| \int_{\Gamma_{\alpha}^{n}} e^{\lambda t} \, R(\lambda) \, x_0 \; d\lambda \right\| \; = \; 0 \qquad\qquad (2.14)$$

and a similar result applies on Γ_{β}^{m} as $m \to -\infty$.

On the left hand side, $\Gamma_{\beta}^{m,n}$ of the rectangle we have $\lambda = \beta + i\nu$ and

$$\|e^{\lambda t} \, R(\lambda) \, x_0\| \; \leq \; e^{-\beta t} \, \|R(\lambda) \, x_0\| \; \leq \; (cf. \;\; (2.10))$$

$$\leq \; e^{-\beta t} \max_{k} |x_k|^2 \sum_{k=-\infty}^{\infty} \frac{1}{(\nu - \nu_k)^2} \; \leq \; N \, e^{-\beta t} \, \|x_0\| \; ,$$

where N is independent of β. Hence we conclude that for $t > 0$

$$\lim_{\beta \to \infty} \int_{\Gamma_\beta^{m,n}} e^{\lambda t} R(\lambda) \, x_0 \, d\lambda = 0. \qquad (2.15)$$

Combining (2.14) and (2.15) we see that, still for $t > 0$,

$$S(t) \, x_0 = \lim_{\substack{m \to -\infty \\ n \to \infty}} \left\{ \lim_{\beta \to -\infty} \frac{1}{2\pi i} \int_{\Gamma_{\alpha,\beta}^{m,n}} e^{\lambda t} R(\lambda) \, x_0 \, d\lambda \right\}$$

$$= \frac{1}{2\pi i} \int_{\Gamma_\alpha} e^{\lambda t} R(\lambda) \, x_0 \, d\lambda \, ,$$

i.e., we can write, for $t > 0$,

$$S(t) = \frac{1}{2\pi i} \int_{\Gamma_\alpha} e^{\lambda t} R(\lambda) \, d\lambda \, ,$$

the integral converging in the strong operator topology.

We complete the present section with a discussion of the convergence in X of the integral (2.04), corresponding to solutions of the inhomogeneous solution with zero initial state, i.e.,

$$x_u(t) = \frac{1}{2\pi i} \int_{\Gamma_\alpha} e^{\lambda t} R(\lambda) \, b \, \hat{u}(\lambda) \, d\lambda.$$

We first of all note that for systems having the properties indicated in the discussion following (2.07) the *Carleson criterion* of [11] is satisfied if the input coefficients ((1.14) ff.) b_k are bounded. In this case the convergence of the indicated integral can be established by contour integration quite similar to that used above in connection with (2.03). There is no need for us to pursue that here, both because the argument would be very similar to the one already given and because the resulting admissibility

criterion would add nothing to what is already obtainable from the Carleson criterion. Instead we will present two examples in which the present convergence criterion is used in a quite different way.

Consider first the almost trivial system

$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} = 0 \ , \ 0 \leq x \leq 1 \ , \ t \geq 0 \ ,$$

whose action is simply that of right translation, with the boundary condition

$$w(0,t) = u(t) \ , \ u \in L^2[0,\infty).$$

This corresponds to a system (1.01) wherein $A = -\frac{\partial}{\partial x}$, a closed unbounded operator on $L^2[0,1]$ with domain consisting of those functions in $H^1[0,1]$ vanishing at $x = 0$. So defined, it is well known that A generates a strongly continuous semigroup. The element b in (1.01) is easily seen to be just $\delta_{\{0\}}$. Traditional spectral analysis does not yield the (almost obvious) conclusion that b is an admissible input element because A has no eigenvectors; indeed, its spectrum is empty.

It is a simple matter to verify that that if we can find the unique solution of (1.01) corresponding to $u(t) \equiv e^{\lambda t}$ in the form $\hat{x}(\lambda)e^{\lambda t}$, $\hat{x}(\lambda) \in X$, then $\hat{x}(\lambda) = R(\lambda)b$. Applied to the present case we see that

$$R(\lambda)b = \hat{w}(x,\lambda) \ ,$$

where $\hat{w}(x,\lambda)$ is the solution of the boundary value problem

$$\lambda\hat{w} + \frac{\partial\hat{w}}{\partial x} = 0, \quad \hat{w}(0,\lambda) = 1 \ .$$

That is, $R(\lambda)b$ is the element of $L^2[0,1]$ given by

$$(R(\lambda)b)(x) = e^{-\lambda x} \ .$$

Accordingly, then, admissibility of $b = \delta_{\{0\}}$ in this case involves the convergence in $L^2[0,1]$ of the integral

$$w_u(x,t) = \frac{1}{2\pi i} \int_{\Gamma_\alpha} e^{\lambda(t-x)} \hat{u}(\lambda) d\lambda \ ,$$

for $\hat{u} \in H^2(\mathbb{C}^+)$, $\alpha > 0$ and Γ_α as described before. The standard theory of the inverse Laplace transform shows that for fixed $t \geq 0$ the integral converges, as a function of x, in $L^2[0,\infty)$, hence in $L^2[0,1]$ in particular, to the function

$$w_u(x,t) = \begin{cases} 0, & x > t \\ u(t-x), & x < t \ . \end{cases}$$

Using the boundedness of the exponential function on Γ_α and its uniform continuity with respect to λ and t on compact sets, an easy application of the Plancherel theorem shows the convergence, in $L^2[0,1]$, to be uniform for t in compact subintervals of $[0,\infty)$. Thus the admissibility of the input element is confirmed here as an immediate consequence of standard Laplace transform theory.

Now we turn to a somewhat less trivial example, involving the partial differential equation

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} = 0 \ , \ t \geq 0, \ 0 \leq x \leq \pi \ , \qquad (2.16)$$

which is of some interest because it is the beginning of under-standing for the far more challenging *Korteweg - deVriess* equation

$$\frac{\partial w}{\partial t} + 3 w \frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3} = 0 \ .$$

We adjoin to (2.16) the boundary conditions

$$\frac{\partial^2 w}{\partial x^2}(0,t) + \alpha\, w(0,t) = u(t) \ , \ u \in L^2[0,\infty) \ , \ |\alpha| \leq 1 \ , \quad (2.17)$$

$$\frac{\partial w}{\partial x}(\pi,t) = 0 \ , \ w(\pi,t) = 0 \ . \quad (2.18)$$

It may be verified that the operator $A = -\dfrac{\partial^3}{\partial x^3}$, with domain consisting of functions in $H^3[0,\pi]$ satisfying boundary conditions which are the homogeneous counterparts of (2.17), (2.18), is dissipative with respect to the norm in $L^2[0,\pi]$, the state space for the semigroup known to be generated by A as a consequence of application of the Lumer - Phillips theorem [14].

Here we compute $R(\lambda)b = \hat{w}(x,\lambda)$, where the latter function is the solution of the boundary value problem

$$\lambda\hat{w} + \frac{\partial^3 \hat{w}}{\partial x^3} = 0 \ , \ t \geq 0, \ 0 \leq x \leq \pi \ , \quad (2.19)$$

$$\frac{\partial^2 \hat{w}}{\partial x^2}(0,\lambda) + \alpha\, \hat{w}(0,\lambda) = 1 \ , \quad (2.20)$$

$$\frac{\partial \hat{w}}{\partial x}(\pi,\lambda) = 0 \ , \ \hat{w}(\pi,\lambda) = 0 \ . \quad (2.21)$$

Setting $\lambda = \mu^3$ and defining $\rho = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, it may be verified that

$$R(\lambda)b = \frac{1}{D(\mu)} \left[\ \mu(\bar{\rho}-\rho)e^{\mu(\bar{\rho}+\rho)\pi}e^{-\mu x} - \mu(\bar{\rho}+1)e^{\mu(\bar{\rho}-1)\pi}e^{\mu\rho x} \right.$$

$$\left. + \ \mu(\rho+1)e^{\mu(\rho-1)\pi}e^{\mu\bar{\rho}x} \ \right] \ , \quad (2.22)$$

where

$$D(\mu) = \mu \left[(\rho+1)e^{\lambda(\rho-1)\pi}(\mu^2\bar{\rho}^2+\alpha)-(\bar{\rho}+1)e^{\mu(\bar{\rho}-1)\pi}(\mu^2\rho^2+\alpha) \right.$$

$$-(\bar{\rho}-\rho)e^{\mu(\bar{\rho}+\rho)\pi}(\mu^2,\alpha)\Big] \ . \tag{2.23}$$

Since the expression on the right hand side of (2.22) is invariant under perturbations on the set -1, ρ, $\bar{\rho}$, it may be seen that $R(\lambda)b$ is, indeed, an analytic function of λ as predicted by the general theory of the resolvent operator. It can be seen that the system eigenvalues (of which we make no use here) are numbers $\lambda_k = \mu_k{}^3$, the μ_k being zeros of $D(\mu)$, asymptotically given by

$$\lambda_k = -\frac{8k^3}{3\sqrt{3}} + \mathcal{O}(k^2) \ .$$

In order to investigate the behavior of $R(\lambda)b$ in the right half λ plane, it is sufficient, taking the symmetry relations already noted into account, to examine the expressions (2.22) and (2.23) in the sector $|\arg \mu| \leq \pi/6$. In that sector the first two exponentials shown in (2.23) are bounded while the third is easily seen to be just $e^{\mu\pi}$. Then, looking at the terms in (2.22) one by one, we can see in the first term that $e^{\mu(\bar{\rho}+\rho)\pi}/D(\mu)$ is uniformly bounded in the indicated μ sector while $e^{-\mu x}$ is uniformly bounded for μ in that sector and $0 \leq x \leq \pi$. In the second and third terms, respectively, $e^{\mu(\bar{\rho}-1)\pi}$ and $e^{\mu(\rho-1)\pi}$ are bounded in the indicated μ sector while $e^{\mu\rho x}/D(\mu)$ and $e^{\mu\bar{\rho}x}/D(\mu)$ are bounded for those values of μ and $0 \leq x \leq \pi$. Then taking the coefficients of the exponentials into account in (2.22) and (2.23) we conclude that the $L^2[0,\pi]$ norm of (2.22) is bounded by some positive multiple of $1/|\mu|^2$ uniformly for $|\arg \mu| \leq \pi/6$. Thus $R(\lambda)b$ has $L^2[0,\pi]$ norm uniformly bounded by a multiple of $|\lambda|^{2/3}$. Since this function is square integrable on Γ_α, Re $\alpha > 0$, we see that (2.04), $R(\lambda)b$ as in (2.22), is convergent in $L^2[0,\pi]$, uniformly in compact subintervals of $t \geq 0$. We conclude then that solutions $w_u(x,t)$ of (2.16), (2.17) and (2.18) are well defined and strongly continuous in $L^2[0,\pi]$, with respect to t. Accordingly, we see that the input mechanism indicated in (2.17) is admissible. We leave it to the interested reader to compute the distributional form of the relevant element b.

3. Admissible Feedback Relations and the Closed Loop Semigroup.

The system (1.01), (1.02) is in "closed loop" operation, with feedback "gain" coefficient $\mathfrak{K}$ if an affine relationship

$$u(t) = \mathfrak{K} \, y(t) + v(t) \qquad (3.01)$$

holds between the actual input, $u(t)$, to the system, the output $y(t)$, and the supplied exogenous input, $v(t)$, the equality holding in the sense appropriate to the spaces in which these functions lie. In terms of the corresponding Laplace transforms we then have

$$\hat{u}(\lambda) = \mathfrak{K} \, \hat{y}(\lambda) + \hat{v}(\lambda). \qquad (3.02)$$

Substituting this relationship into (1.07) we have

$$\hat{y}(\lambda) = T(\lambda) \left[\mathfrak{K} \, \hat{y}(\lambda) + \hat{v}(\lambda) \right],$$

so that now

$$\hat{y}(\lambda) = \frac{T(\lambda)}{1 - \mathfrak{K} \, T(\lambda)} \, \hat{v}(\lambda) \equiv T_{\mathfrak{K}}(\lambda) \, \hat{v}(\lambda). \qquad (3.03)$$

We will refer to $T_{\mathfrak{K}}(\lambda)$ as the *closed loop transfer function* corresponding to the gain coefficient $\mathfrak{K}$ or, more briefly, as the $\mathfrak{K}$-*transfer function*. We will say that $\mathfrak{K}$ is an *admissible gain coefficient* if there is a real number $a_{\mathfrak{K}}$ such that $T_{\mathfrak{K}}(\lambda)$ is holomorphic and uniformly bounded for Re $\lambda \geq a_{\mathfrak{K}}$. Along the same lines, $\mathfrak{K}$ is a *totally admissible gain coefficient* if it is admissible, as just described, and the resulting closed loop system is well posed in the sense familiar in the semigroup framework. One must not neglect to leave open the possibility that some non zero value of $\mathfrak{K}$ might be admissible in one of these senses while $\mathfrak{K}=0$, corresponding to the original open loop system, is not. That is, input and output elements b and c may well be jointly admissible only for $\mathfrak{K}$ in some subset of R^1, and that subset need not necessarily include $\mathfrak{K} = 0$. As a consequence we see that feedback transformations on an infinite dimensional linear input - output system do not necessarily form a group as in the finite dimensional case discussed in [24].

The linear transformation defined for x in the domain of A and with range in the space of linear functionals defined on the domain of A* given by

$$< x, c > b$$

will be denoted by $bc^* x$. This operator plays an important role in connection with the $\mathfrak{K}$ - *closed loop system*

$$\dot{x} = A x + (\mathfrak{K} y) b + b v , \qquad (3.04)$$

$$y = c^* x . \qquad (3.05)$$

We will say that $\mathfrak{K}$ is a *generating gain coefficient* if the closed-loop system just described corresponds to a strongly continuous semigroup, which we can then show to be equivalent to the system

$$\dot{x} = A_{\mathfrak{K}} x = \left[A + \mathfrak{K} bc^* \right] x , \qquad (3.06)$$

at the same time giving an interpretation of the operator $A_{\mathfrak{K}}$.

Before stating the next result let us observe that, from the assumed properties of $T(\lambda)$, we may identify its inverse transform with a distribution μ, called the *impulse response distribution*, for for which the associated convolution operation may be described by

$$y(t) = \int_0^t u(s) \, d \, \mu(t-s) . \qquad (3.07)$$

Then let us note that, for $\mathfrak{K} \neq 0$,

$$T_{\mathfrak{K}}(\lambda) = \frac{T(\lambda)}{1-\mathfrak{K}T(\lambda)} = \frac{1}{\mathfrak{K}} \frac{(\mathfrak{K}T(\lambda)-1) + 1}{1 -\mathfrak{K}T(\lambda)} =$$

$$= \frac{1}{\mathfrak{K}} \left[\frac{1}{1 -\mathfrak{K}T(\lambda)} - 1 \right] \equiv \frac{1}{\mathfrak{K}} \left[U_{\mathfrak{K}}(\lambda) - 1 \right] . \qquad (3.08)$$

From this it is clear that $T_{\mathfrak{K}}(\lambda)$ is holomorphic and bounded for

Re $\lambda > a_\mathfrak{K}$ if and only this is also true of $U_\mathfrak{K}(\lambda)$. The latter property, via the transformed equation

$$\hat{u}(\lambda) = U_\mathfrak{K}(\lambda) \ \hat{v}(\lambda) \ , \quad \text{equiv.:} \quad \hat{u}(\lambda) = \mathfrak{K} \ T(\lambda) \ \hat{u}(\lambda) + \hat{v}(\lambda), \quad (3.09)$$

may be seen to be equivalent to the solvability of the convolution equation $u = \mathfrak{K} \ \mu \ \# \ u + v$, '#' standing for the convolution operation (thus (3.07) could also be written as $y = \mu \ \# \ u$) , that is, of

$$u(t) \quad = \quad \mathfrak{K} \int_0^t u(s) \ d \ \mu(t-s) + v(t) \quad\quad (3.10)$$

in the L^2 sense; i.e., if the inhomogeneous term v is in $L^2_\beta[0,\infty)$ then there exists a unique solution u lying in $L^2_\gamma[0,\infty)$, where $\gamma = $ max $\{\alpha,\beta\}$, for every $\alpha > a_\mathfrak{K}$. A further property of $U_\mathfrak{K}(\lambda)$, which will be developed more fully in the proof of Theorem 3.1 below and in the work of §4 to follow, is that

$$\hat{y}(\lambda) = U_\mathfrak{K}(\lambda) \ \hat{y}_0(\lambda), \quad y_0(t) = S(t) \ x_0 \ ,$$

describes, in the frequency domain, the mapping from the output obtained from the open loop homogeneous system (2.01) with initial state x_0 , via (1.02), to the output obtained from the closed loop system (3.06), with feedback gain $\mathfrak{K}$, via the same output relation.

With these preliminaries taken care of, we may state

<u>Theorem 3.1</u>. *If* b *and* c *are totally admissible for the system* (1.01), (1.02), *and if* $\mathfrak{K}$ *is an admissible gain coefficient for that system, then they are totally admissible for the closed loop system* (3.04), (3.05).

<u>Proof</u>. Let $v \in L^2_\beta[0,\infty)$, that space being defined as in §1, and let u be the solution in $L^2_\gamma[0,\infty)$, where γ is now max $\{\alpha,\beta\}$ for some $\alpha >$

a_κ, of the convolution equation (3.10); an alternative approach consists in letting $\hat{u}(\lambda)$ be defined in terms of $\hat{v}(\lambda)$ by means of the Laplace transform relation (3.09). Since b and c are totally admissible for (1.01),(1.02), the input u to that system yields the output y with $\hat{y}(\lambda) = T(\lambda)\hat{u}(\lambda)$ (equivalently, (3.07)). Then, since u obeys (3.01), we conclude that the solution of (1.01) with the input u(t) must be the same as that of (3.04) with the input v(t); thus y(t), as a consequence, is also the output from (3.04) via (3.05). Since v is an arbitrary element of $L_\beta^2[0,\infty)$ and the solution of (3.04), being the same as that of (1.01) with the input u, has the same properties as before, we conclude that b is also an admissible input element for the system (3.04), (3.05).

Now consider the homogeneous counterpart of (3.04):

$$\dot{x} = A x + \kappa b y \tag{3.11}$$

with the same output relation (3.05). Clearly, if we let y(t) be the solution of the convolution equation describing the open loop output to closed loop output, i.e.

$$y(t) = c^* e^{At} x_0 + \kappa \int_0^t y(s) \, d\, \mu(t-s) , \tag{3.12}$$

the admissibility of c as an output element (cf. (1.06)) shows that

$$y_0(t) = c^* e^{At} x_0$$

lies in $L_\alpha^2[0,\infty)$. Then the solution y(t) of (3.12), by the same argument used for u(t) in connection with (3.10), lies in $L_\gamma^2[0,\infty)$, $\gamma = \max\{\alpha,\delta\}$, $\delta > a_\kappa$, so c is an admissible output element for (3.10).

When the feedback relation (3.01) is applied with $v(t) \equiv 0$ the resulting trajectory, x(t), and the output, y(t), depend in a linear homogeneous manner on the initial state x_0. Indeed, as we shall see, there is a strongly continuous semigroup $S_\kappa(t)$ of bounded linear operators on X such that

$$x(t) = S_\Re(t) \, x_0 \, . \qquad\qquad (3.13)$$

In this section we will establish the existence of $S_\Re(t)$, following a procedure much the same as that presented in [19], we will examine some of its important properties and we will discuss its generating operator $A_\Re$. We begin with the statement of

Theorem 3.2 *Let b and c be totally admissible for the system* (1.01), (1.02) *and let* $\Re$ *be an admissible gain coefficient. Let x(t) be the solution of* (3.04),(3.05) *corresponding to an initial state* $x_0 \in X$. *Then there is a strongly continuous semigroup of operators,* $S_\Re(t)$ *such that x(t) is given by* (3.13). *Moreover, with appropriate in pretation of domain, the generator of* $S_\Re(t)$ *is the operator* $A_\Re$ *shown in* (3.06).

Proof. We denote by S(t) the strongly continuous semigroup of bounded operators on X generated by the operator A, i.e., $S(t) = e^{At}$, for $t \geq 0$. Since c is an admissible output element, making the same growth assumption on S(t) as made in Theorem 1.1 in §1, we conclude that

$$y_0(t) \equiv c^* \, S(t) \, x_0 \in L_\alpha^2[0,\infty) \, , \quad \alpha > a \, .$$

From the transfer function criterion for joint admissibility of $\Re$ we may solve for $y(t)$ in

$$(V_\Re \, y)(t) \equiv y(t) - \Re \int_0^t y(s) \, d \, \mu(t-s) = y_0(t) \qquad (3.14)$$

to obtain the closed loop output corresponding to the initial state x_0. Since the closed loop input is given by

$$u(t) = \Re \, y(t) \, ,$$

as is already anticipated in (3.14), the familiar "variation of parameters" formula gives

$$x(t) = S(t) x_0 + \mathcal{R} \int_0^t S(t-s) \, b \, y(s) \, ds \ . \qquad (3.15)$$

The right hand side of (3.15) is, in fact, $S_{\mathcal{R}}(t) x_0$, but it is a little bit awkward to display the form of this operator because it involves the inverse of the Volterra - type operator $V_{\mathcal{R}}$ appearing in (3.14); specifically, we have

$$y(t) = V_{\mathcal{R}}^{-1} c^* S(t) x_0 \qquad (3.16)$$

and so

$$x(t) = \left\{ S(t) + \mathcal{R} \int_0^t S(t-s) \, b \, V_{\mathcal{R}}^{-1} c^* S(s) \, ds \right\} x_0 \equiv S_{\mathcal{R}}(t) x_0. \quad (3.17)$$

To demonstrate the strong continuity, we note that the admissibility of c shows that y_0 is locally square integrable. The solvability of (3.12) in $L^2[0,T]$, $T > 0$, shows that $y(t)$ is locally square integrable and then the admissibility of the input element b shows that $x(t)$, as defined by (3.16), is continuous in t. Anticipating our proof, in the sequel, that there is a semigroup $S_{\mathcal{R}}(t)$ such that $x(t) = S_{\mathcal{R}}(t) x_0$, for x_0 an arbitrary element of X, the strong continuity of $S_{\mathcal{R}}(t)$ with respect to t follows. Since, as shown in [11], the norm of

$$\hat{x}(t) = \int_0^t S(t-s) \, b \, u(s) \, ds \ ,$$

b an admissible input element, can be bounded in terms of $\|u\|_{L^2[0,t]}$, with uniform bound for any finite range of t, it easily follows that

$$\lim_{t \downarrow 0} x(t) = 0 \ .$$

We will subsequently indicate another way in which the strong continuity property can be established.

The Laplace transform of the operator $S_{\mathcal{R}}(t)$, which we denote by $R_{\mathcal{R}}(\lambda)$, is somewhat more symmetric and more tractible in computations. The corresponding transform of $S(t)$ is well known ([8]) to be the resolvent operator $R(\lambda) = (\lambda I - A)^{-1}$. Taking the transform of of (3.15) we have

$$\hat{x}(\lambda) = R(\lambda)\, x_0 + \mathcal{R}\, R(\lambda)\, b\, \hat{y}(\lambda)$$

and then, since (cf. (3.16)) $\hat{y}(\lambda) = U_{\mathcal{R}}(\lambda)\, c^*\, R(\lambda)\, x_0$ (recall that $U_{\mathcal{R}}(\lambda) \equiv (1 - \mathcal{R}\, T(\lambda))^{-1}$) , we have

$$\hat{x}(\lambda) = \left\{ R(\lambda) + \mathcal{R}\, R(\lambda)\, b\, U_{\mathcal{R}}(\lambda)\, c^*\, R(\lambda) \right\} x_0$$

and consequently the formula

$$R_{\mathcal{R}}(\lambda) = R(\lambda) + \mathcal{R}\, R(\lambda)\, b\, U_{\mathcal{R}}(\lambda)\, c^*\, R(\lambda) \,. \qquad (3.18)$$

Since it is equally true that $y(t) = c^* S_{\mathcal{R}}(t) x_0$, it is also true that

$$R_{\mathcal{R}}(\lambda) = R(\lambda) + \mathcal{R}\, R(\lambda)\, bc^*\, R_{\mathcal{R}}(\lambda) = R(\lambda) \left[I + \mathcal{R}\, bc^*\, R_{\mathcal{R}}(\lambda) \right], \quad (3.19)$$

which we save for later reference.

We remark, since we have already seen that $S_{\mathcal{R}}(t)$ is a strongly continuous semigroup, that $R_{\mathcal{R}}(\lambda)$ must be the resolvent, $(\lambda I - A_{\mathcal{R}})^{-1}$, of the closed loop generating operator $A_{\mathcal{R}}$ for $S_{\mathcal{R}}(t)$ and may, in consequence, be expected to yield important information about the form of that operator and the meaning of its expression in (3.06).

Finally, we must establish the semigroup property of $S_{\mathcal{R}}(t)$. To this end we will make use of a lemma, stated and proved at the end of the proof of the present theorem, to the effect that $R_{\mathcal{R}}(\lambda)$ satisfies

the resolvent identity

$$R_{\mathfrak{K}}(\lambda)\ R_{\mathfrak{K}}(\mu)\ =\ \frac{1}{\mu - \lambda}\ \left[\ R_{\mathfrak{K}}(\lambda)\ -\ R_{\mathfrak{K}}(\mu)\ \right]\ . \qquad (3.20)$$

Assuming this, we prove the semigroup property of $S_{\mathfrak{K}}(t)$ in much the same way as a parallel demonstration is carried out by Kato ([12]). We let T_1 and T_2 be positive numbers and we let $\alpha_1 > \alpha_2 > a_{\mathfrak{K}}$, the last as defined above for the admissible feedback coefficient $\mathfrak{K}$. We let Γ_{α_1} and Γ_{α_2} be corresponding contours whose definition is the same as that of Γ_{α} in the preceding section. Then we have

$$S_{\mathfrak{K}}(T_1)\ S_{\mathfrak{K}}(T_2)\ x_0\ =\ \frac{1}{2\pi i}\int_{\Gamma_{\alpha_2}} e^{\lambda T_2}\ R_{\mathfrak{K}}(\lambda)\ S_{\mathfrak{K}}(T_1)\ x_0\ d\lambda$$

$$=\ -\ \frac{1}{4\pi^2}\int_{\Gamma_{\alpha_2}} e^{\lambda T_2}\ R_{\mathfrak{K}}(\lambda)\int_{\Gamma_{\alpha_1}} e^{\mu T_1}\ R_{\mathfrak{K}}(\mu)\ x_0\ d\mu\ d\lambda$$

$$=\ -\ \frac{1}{4\pi^2}\int_{\Gamma_{\alpha_2}}\int_{\Gamma_{\alpha_1}} e^{\lambda T_2 + \mu T_1}\ R_{\mathfrak{K}}(\lambda)\ R_{\mathfrak{K}}(\mu)\ x_0\ d\mu\ d\lambda\ =\ (\text{using }(3.20))$$

$$=\ -\ \frac{1}{4\pi^2}\int_{\Gamma_{\alpha_2}}\int_{\Gamma_{\alpha_1}} e^{\lambda T_2 + \mu T_1}\ \frac{1}{\mu - \lambda}\ \left[\ R_{\mathfrak{K}}(\lambda)\ -\ R_{\mathfrak{K}}(\mu)\ \right]\ x_0\ d\mu\ d\lambda$$

$$=\ -\ \frac{1}{4\pi^2}\int_{\Gamma_{\alpha_2}} e^{\lambda T_2}\ R_{\mathfrak{K}}(\lambda)\ x_0\int_{\Gamma_{\alpha_1}} \frac{e^{\mu T_1}}{\mu - \lambda}\ d\mu\ d\lambda$$

$$+\ \frac{1}{4\pi^2}\int_{\Gamma_{\alpha_2}} e^{\lambda T_2}\int_{\Gamma_{\alpha_1}} \frac{e^{\mu T_1}}{\mu - \lambda}\ R_{\mathfrak{K}}(\mu)\ x_0\ d\mu\ d\lambda\ .$$

In the first integral, since $\text{Re}\ \mu > \text{Re}\ \lambda$, we have

$$\int_{\Gamma_{\alpha_1}} \frac{e^{\mu T_1}}{\mu - \lambda} \, d\mu = 2\pi i \, e^{\lambda T_1} \, .$$

Changing the order of integration in the second integral we have

$$- \frac{1}{4\pi^2} \int_{\Gamma_{\alpha_1}} e^{\mu T_1} \, R_{\mathfrak{K}}(\mu) \, x_0 \int_{\Gamma_{\alpha_2}} \frac{e^{\lambda T_2}}{\mu - \lambda} \, d\lambda \, d\mu \, .$$

But, since $\mathrm{Re} \, \mu > \mathrm{Re} \, \lambda$, $e^{\lambda T_2}$ and $\frac{1}{\mu - \lambda}$ are both holomorphic in the closed half plane $\mathrm{Re} \, \lambda \lesssim \alpha_2$ and the familiar Jordan lemma argument may be applied to show that the integral is zero. Thus we have

$$S_{\mathfrak{K}}(T_1) \, S_{\mathfrak{K}}(T_2) \, x_0 = \frac{1}{2\pi i} \int_{\Gamma_{\alpha_2}} e^{\lambda T_2 + \lambda T_1} \, R_{\mathfrak{K}}(\lambda) \, x_0 \, d\lambda = S_{\mathfrak{K}}(T_1 + T_2) \, x_0$$

and the proof of the semigroup property is complete, pending the proof of the lemma which follows immediately.

<u>Lemma 3.3</u>   *Let μ and λ be distinct complex numbers with real parts $> a_{\mathfrak{K}}$. Then $R_{\mathfrak{K}}(\lambda)$ satisfies the resolvent identity* (3.20).

<u>Proof</u>. First of all we remind ourselves that $R(\lambda)$, being the resolvent of the operator A, must satisfy its version of the identity:

$$\frac{1}{\mu - \lambda} \left[R(\lambda) - R(\mu) \right] = R(\mu) \, R(\lambda). \tag{3.21}$$

Then, using (3.18), we compute

$$R_{\mathfrak{K}}(\lambda) \, R_{\mathfrak{K}}(\mu) = R(\lambda) \, R(\mu) + \mathfrak{K} \, U_{\mathfrak{K}}(\lambda) \, R(\lambda) \, bc^* \, R(\lambda) \, R(\mu) +$$

$$+ \mathfrak{K} \, U_{\mathfrak{K}}(\mu) \, R(\lambda) \, R(\mu) bc^* R(\mu) + \mathfrak{K}^2 \, U_{\mathfrak{K}}(\lambda) U_{\mathfrak{K}}(\mu) \, R(\lambda) bc^* R(\lambda) \, R(\mu) bc^* R(\mu) =$$

(using the resolvent identity (3.21) in each term and rearranging)

$$= \frac{1}{\mu - \lambda} \left[R(\lambda) - R(\mu) + \Re\, U_{\Re}(\lambda)\, R(\lambda)\, bc^* R(\lambda) - \Re\, U_{\Re}(\mu)\, R(\mu)\, bc^* R(\mu) \right.$$

$$\left. + \left\{ \Re^2 (T(\lambda) - T(\mu)) U_{\Re}(\lambda) U_{\Re}(\mu) - \Re\, U_{\Re}(\lambda) + \Re\, U_{\Re}(\mu) \right\} R(\lambda)\, bc^* R(\mu) \right].$$

Using the formulae (1.08) and (3.08) for $T(\lambda)$ and $U_{\Re}(\lambda)$, the term in brackets $\{,\}$ is easily seen to be zero. Then, noting (3.18) again, the result follows.

We remark that the semigroup $S_{\Re}(t)$ can also be represented by

$$S_{\Re}(t)\, x_0 = \frac{1}{2\pi i} \int_{\Gamma_\alpha} e^{\lambda t}\, R_{\Re}(\lambda)\, d\lambda\, x_0 =$$

$$= \frac{1}{2\pi i} \int_{\Gamma_\alpha} e^{\lambda t} \left[R(\lambda) + \Re\, R(\lambda)\, b\, U_{\Re}(\lambda)\, c^* R(\lambda) \right] x_0\, d\lambda. \quad (3.22)$$

The integral converges in the topology of the space X; the first term following the discussion at the end of §2, the second from the uniform boundedness of $U_{\Re}(\lambda)$ for $\mathrm{Re}\ \lambda \geq \alpha > a_{\Re}$ and the fact that the admissibility of the input element c guarantees that $c^* R(\lambda)\, x_0 \in H^2(\mathbb{C}^+)$ and our discussion, also in §2, of the integral (2.02). Once we have the convergent representation (3.22), the strong continuity of $S_{\Re}(t)$ with respect to t follows by replacing $e^{\lambda t}$ in (3.22) by the difference $e^{\lambda t_1} - e^{\lambda t_2}$ and treating the resulting integral by the method already used in connection with (2.02) in §2, thereby providing a proof of that property independent from the one used earlier in the proof of Theorem 3.2.

The strongly continuous semigroup $S_{\Re}(t)$ has (see, e.g.,[8]) a closed generator $A_{\Re}$ whose resolvent is the operator function $R_{\Re}(\lambda)$ discussed above. We proceed now to use the form of $R_{\Re}(\lambda)$ and its

properties to identify the operator A_R and its domain, $\mathcal{D}(A_R)$. In order to do this we have to make some further assumptions about the original operator, A, and its relationship to the spaces X, W, W' introduced in §1. We have already assumed there that $\mathcal{D}(A) \subset W$ which is densely and continuously imbedded in X. We now further suppose that $\mathcal{D}(A)$ is a closed subspace of W defined by a finite number of linear equations:

$$\mathcal{D}(A) = \left\{ w \in W \mid d_k^* w = 0, \; k = 1, 2, \ldots, n \right\}, \qquad (3.23)$$

where $d_k \in W'$ for each k and, following the convention introduced earlier, $d_k^* w$ is a convenient way to represent $<w, d_k>$. Moreover, we recall that $b \in W'$ and we further assume:

i) The domain of $R(\lambda)$ can be extended from X to a closed subspace of W', which we will denote by $\mathcal{D}(R)$, which is independent of λ and includes the admissible input element b. So extended, we have

$$R(\lambda) : \mathcal{D}(R) \rightarrow X$$

for each λ with Re $\lambda > a$.

ii) $R(\lambda) b \in W - \mathcal{D}(A)$.

We remark that condition (i) is an agreement with a characterization of admissible input elements (operators, in fact) obtained by Weiss in [22].

Redefining the d_k appropriately through formation of linear combinations of the original elements, we may assume that there is just one of them, $d_n = d_n(\lambda)$ (the new d_k may depend on λ), such that

$$d_n^* R(\lambda) b = 1 \qquad (3.24)$$

while

$$d_k^* \, R(\lambda) \, b \;=\; 0, \quad k = 1, 2, \ldots, n-1.$$

From the formula (3.18) for $R_{\mathcal{K}}(\lambda)$ and condition ii) above we conclude that for each $x \in X$

$$w \;=\; R_{\mathcal{K}}(\lambda) \, x \in W \; .$$

Then, using (3.18) again, we have

$$w = R(\lambda) \, x + \mathcal{K} \, R(\lambda) \, bc^* \, R_{\mathcal{K}}(\lambda) \, x = R(\lambda) \, x + \mathcal{K} \, R(\lambda) \, bc^* \, w \; . \tag{3.25}$$

Since $R(\lambda) : X \to \mathcal{D}(A) \subset W$, (3.23) and (3.25) yield

$$d_n^* w \;=\; d_n^* \, R(\lambda) \, x + \mathcal{K} \, d_n^* \, R(\lambda) \, bc^* \, w \;=\; 0 + \mathcal{K} \, c^* w \; . \tag{3.26}$$

We similarly find that $d_k^* w = 0$, $k = 1, 2, \ldots, n-1$. It follows that the range of $R_{\mathcal{K}}(\lambda)$, which is the domain of $A_{\mathcal{K}}$, can be described by

$$\mathcal{D}(A_{\mathcal{K}}) = \left\{ \, w \in W \; \middle| \; d_k^* w = 0, \; k = 1, 2, \ldots, n-1, \; d_n^* w = \mathcal{K} \, c^* \, w \, \right\} \; .$$

If we agree to extend the definition of A beyond the original $\mathcal{D}(A)$ so that

$$(\lambda I - A) \, R(\lambda) \, b = b, \quad \text{i.e.,} \quad A \, R(\lambda) \, b = \lambda \, R(\lambda) \, b - b \; ,$$

then we find that, with $x = R(\lambda) \, w$,

$$(\lambda I - A) \, w = x + \mathcal{K} \, bc^* \, w \tag{3.27}$$

so that, in this sense, it is true that

$$\left[\, \lambda I - (A + \mathcal{K} \, bc^*) \, \right] w \;=\; x$$

and thus, with the domain of A in W so extended, we see that $R_{\mathcal{K}}(\lambda)$ is the resolvent of

$$A_{\mathcal{R}} = A + \mathcal{R} bc^* .$$

To check the consistency of this, we note that if $w \in \mathcal{D}(A_{\mathcal{R}})$, as just described above, then (3.27) is true and, since

$$c^* R(\lambda) x = \mathcal{R}^{-1} d_n{}^* R(\lambda) x = 0 ,$$

we have

$$A_{\mathcal{R}} w = A R(\lambda) x + \mathcal{R} A_{\mathcal{R}} R(\lambda) bc^* w$$

$$= A R(\lambda) x + \mathcal{R} \left[A R(\lambda) bc^* w + \mathcal{R} bc^* R(\lambda) bc^* w \right]$$

$$= A R(\lambda) x + \mathcal{R} \left[\lambda R(\lambda) b - b + b \mathcal{R} \mathcal{R}^{-1} d_n{}^* R(\lambda) b \right] c^* w$$

$$= A R(\lambda) x + \mathcal{R} \lambda R(\lambda) bc^* w \in X$$

and so we see that

$$A_{\mathcal{R}} : \mathcal{D}(A_{\mathcal{R}}) \subset W \rightarrow X$$

as we should expect.

While we have used the representation (3.18) of $R_{\mathcal{R}}(\lambda)$ here under the assumption that b and c are admissible input and output elements, respectively, and also that they are jointly admissible, actually this representation remains useful for the discussion of systems subject to linear output feedback even when some of these assumptions are not satisfied, provided the integral (3.22) remains appropriately convergent for each initial state $x_0 \in X$. In the next section we provide a significant example of a situation of this type. However, an additional difficulty arises here in showing that $S_{\mathcal{R}}(t)$ is strongly continuous because we are not able to use the admissibility of b and c along with (3.15) to establish that property as we have in the argument above.

We will say that the closed loop semigroup $S_{\mathcal{R}}(t)$ is *formally*

defined via $R_\kappa(\lambda)$ if the formula (3.18) is valid, with appropriate interpretation of the products involved, and yields a uniformly bounded analytic operator valued function of λ in some right half plane Re $\lambda > a_\kappa$ for which the resolvent identity can be established. Then we have

Theorem 3.4. *Suppose:*

(i) $S_\kappa(t)$ *is formally defined via* $R_\kappa(\lambda)$;

(ii) *The integral (3.22) remains strongly convergent, in the sense already described above, to a bounded operator* $S_\kappa(t)$;

(iii) *There is a dense subset* $\mathbf{D} \subset X$ *such that for each initial state* $x_0 \in \mathbf{D}$ *a solution* $x_\kappa(t)$ *of (3.06) is defined which is continuous in* t *for* $t \geq 0$ *and is given by the right hand side of* (3.22), *i.e.,* $x_\kappa(t) = S_\kappa(t)$.

Then $S_\kappa(t)$ *is a strongly continuous semigroup of bounded operators for* $t \geq 0$.

The proof is too routine to require detailed presentation here. It will be used in a nontrivial way, particularly in regard to the condition (iii), in a significant example to be presented in the next section which includes, for all practical purposes, a proof of Theorem 3.4.

4. A "Non-Standard" Case Study.

Because of presumed applications to vibration stabilization in flexible space structures, a variety of partial differential equations, and associated boundary conditions, modelling the motion of elastic beams have been studied over the past few years. Here we study certain aspects of the simplest, the so-called *Euler - Bernoulli*, model. We will suppose that units have been normalized so that the equation can be written in the form

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = 0 \ , \ 0 < x < \pi \ , \tag{4.01}$$

and we will study the case wherein the beam is simply supported at each end with a torque $u(t) \in L^2[0,\infty)$ acting at the end $x = \pi$:

$$w(0,t) = 0 \ , \ \frac{\partial^2 w}{\partial x^2}(0,t) = 0 \ , \ w(\pi,t) = 0, \ \frac{\partial^2 w}{\partial x^2}(\pi,t) = u(t). \tag{4.02}$$

We will suppose that an observation, or output, $y(t)$, is available via

$$y(t) = \frac{\partial^2 w}{\partial t \partial x}(\pi,t) \ . \tag{4.03}$$

The associated first order system involves a state

$$\begin{bmatrix} w(x,t) \\ v(x,t) \end{bmatrix} \ , \ v(x,t) = \frac{\partial w}{\partial t}(x,t) \ ,$$

and it is appropriate to take the state space to be

$$X = H^2_{0,0}[0,\pi] \times L^2[0,\pi] \ , \tag{4.04}$$

where the subscripts indicate that we impose on the first component the boundary conditions

$$w(0) = w(\pi) = 0.$$

We take W to be the space

$$W = H_{0,0}^4[0,\pi] \times H_{0,0}^2[0,\pi],$$

which includes $\mathfrak{D}(A)$, where A is the generating operator

$$A = \begin{bmatrix} 0 & I \\ -D^4 & 0 \end{bmatrix}.$$

We will use the inner product consistent with system energy:

$$\left(\begin{bmatrix} w_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ v_2 \end{bmatrix}\right)_X = (w_1'', w_2'')_{L^2[0,\pi]} + (v_1, v_2)_{L^2[0,\pi]}, \quad (4.05)$$

which is equivalent to the standard (Sobolev) inner product in X as defined in [1],e.g.. The sesquilinear functional relation $< z, z' >$ between elements $z \in W$ and $z' \in W'$ is defined so as to reduce to this inner product whenever $z' \in X \subset W'$. We can readily observe that $\mathfrak{D}(A)$ is the closed subspace of W characterized by homogeneous boundary conditions on w and v of the form

$$w(0) = w''(0) = w(\pi) = w''(\pi) = v(0) = v(\pi) = 0.$$

It can be seen, with the use of methods similar to those developed in [11], that the input and output elements corresponding to (4.02) and (4.03) are

$$b = c = \begin{bmatrix} 0, \\ \delta_{\frac{\pi}{2}} \end{bmatrix}. \quad (4.06)$$

This, or a nearly equivalent, system has been cited by a number of researchers (see, e.g., [19]) as an elementary example of physical interest for which neither the input element nor the output element is admissible in the sense of our earlier definition. We will also see that b and c are not jointly admissible in this case for any value of κ (cf. §3).

To compute $T(\lambda)$ and $R(\lambda)b$ we assume an input $u(t) = e^{\lambda t}$, for some complex number λ, and we look for a solution of (4.01), (4.02)

of the form $w(x,t) = e^{\lambda t} \hat{w}(x,\lambda)$, leading to replacement of (4.01) by

$$\lambda^2 \hat{w} + \frac{\partial^4 \hat{w}}{\partial x^4} = 0 \qquad (4.07)$$

and the boundary conditions (4.02) by

$$\hat{w}(0,\lambda) = 0, \quad \frac{\partial^2 \hat{w}}{\partial x^2}(0,\lambda) = 0 \ , \quad \hat{w}(\pi,\lambda) = 0, \quad \frac{\partial^2 \hat{w}}{\partial x^2}(\pi,\lambda) = 0 \ . \quad (4.08)$$

The observation becomes $y(t) = e^{\lambda t} \hat{y}(\lambda)$ with

$$\hat{y}(\lambda) = \lambda \frac{\partial \hat{w}}{\partial x}(\pi,\lambda) \ . \qquad (4.09)$$

Setting $\lambda = i\omega^2$ we can readily compute that

$$\hat{w}(x,i\omega^2) = \frac{1}{2\omega^2} \left[\frac{\sinh \omega x}{\sinh \omega \pi} - \frac{\sin \omega x}{\sin \omega \pi} \right] \qquad (4.10)$$

when ω is not a real or imaginary integer. When $\lambda = 0$ we have

$$\hat{w}(x,0) = \frac{1}{6\pi} x^3 - \frac{\pi}{6} x \ .$$

The observation being $y(t) = e^{\lambda t} \hat{y}(\lambda)$, where (cf. (4.09))

$$\hat{y}(i\omega^2) = \frac{i\omega}{2} \left[\coth \omega \pi - \cot \omega \pi \right] \ .$$

we conclude that this is also the transfer function for the system;

$$T(\lambda) = T(i\omega^2) = \hat{y}(i\omega^2) = \frac{i\omega}{2} \left[\coth \omega \pi - \cot \omega \pi \right] \ , \qquad (4.11)$$

the poles of which are easily seen to lie on the imaginary axis of the λ plane at $\lambda = \pm ik^2$, $k = 1,2, \ldots$. We may examine the behavior of this transfer function on vertical lines $\lambda = a + i\nu$, $-\infty < \nu < \infty$, by looking at $T(i\omega^2)$ on the hyperbolic curves

$$(\omega = \rho + i \sigma), \quad \sigma = - \frac{a}{2\rho} , \tag{4.12}$$

lying in the second and fourth quadrants of the complex ω plane.

From the formula (4.11) it is clear that $T(i\omega^2)$ does not remain bounded on curves (4.12) as $|\omega| \to \infty$, so we conclude that b and c, as specified above, are not jointly admissible for $\hat{K} = 0$, i.e., for the open loop system. Since $e^{\lambda t}R(\lambda)b$ is the exponential solution (expressed as an element of the state space (4.04) corresponding to the input $u(t) = e^{\lambda t}$, we can use (4.10), and its special case corresponding to $\omega = 0$, as we have already noted, to compute

$$R(i\omega^2) \ b = \begin{bmatrix} \hat{w}(x,i\omega^2) \\ i\omega^2\hat{w}(x,i\omega^2) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \omega^{-2} \\ i \end{bmatrix} \begin{bmatrix} \frac{\sinh \omega x}{\sinh \omega \pi} - \frac{\sin \omega x}{\sin \omega \pi} \end{bmatrix}. \tag{4.13}$$

As $|\omega| \to \infty$ along curves of the form (4.12), the real and imaginary poles of (4.13) are approached at a distance $\mathcal{O}(|\omega|^{-1})$ and, from this it can be seen that (4.13) does not remain bounded on such curves, hence $R(\lambda) \ b$ is not bounded on any of the lines $\text{Re } \lambda = a, a > 0$. But we have seen, in §2, that such boundedness, for a sufficiently large, is a necessary condition for admissibility of the input element b. As a result we conclude that b, as described by (4.06), is not an admissible input element for the system (4.01), (4.02). We similarly conclude that $c = b$ is not an admissible output element.

Next we want to study the form of $c^* R(i\omega^2) = b^* R(i\omega^2)$. For $z_0 \in X$, this is given by the formula

$$b^* R(i\omega^2) \ z_0 = \int_0^\infty e^{-i\omega^2 t} < z_0, \ S(t)^* \ b > dt$$

= (since A is antihermitian relative to the inner product (4.05))

$$= < z_0, \int_0^\infty e^{i\bar{\omega}^2 t} \ S(-t) \ b \ dt > \quad = \quad < z_0, \ (-i\bar{\omega}^2 I + A)^{-1} \ b >$$

$$= - \langle z_0, (i\bar{\omega}^2 I - A)^{-1} b \rangle \ .$$

Taking the formula (4.13) and the form of the inner product into account, we conclude that, expanding z_0 now as $\begin{bmatrix} w_0 \\ v_0 \end{bmatrix}$,

$$b^* R(i\omega^2) \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} = i\omega^2 \int_0^\pi \hat{w}(x, -i\omega^2) v_0(x) \ dx - \int_0^\pi \frac{\partial^2}{\partial x^2} \hat{w}(x, -i\omega^2) w_0''(x) \ dx$$

$=$ (since $-i\omega^2 = i(i\omega)^2$ and the formula (4.10) shows that $\hat{w}$ remains invariant when ω is replaced by $i\omega$)

$$= i\omega^2 \int_0^\pi \hat{w}(x, i\omega^2) v_0(x) \ dx - \int_0^\pi \frac{\partial^2}{\partial x^2} \hat{w}(x, i\omega^2) w_0''(x) \ dx$$

$$= \frac{i}{2} \int_0^\pi \left[\frac{\sinh \omega x}{\sinh \omega \pi} - \frac{\sin \omega x}{\sin \omega \pi} \right] v_0(x) \ dx$$

$$- \frac{1}{2} \int_0^\pi \left[\frac{\sinh \omega x}{\sinh \omega \pi} + \frac{\sin \omega x}{\sin \omega \pi} \right] w_0''(x) \ dx \ .$$

Thus we can see that that the resolvent difference used in Theorem 3.2 here becomes the operator valued function defined by

$$-\hat{R} \ R(i\omega^2) \ bb^* \ R(i\omega^2) \ U_{\hat{R}}(i\omega^2) \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} =$$

$$-\hat{R} \int_0^\pi \hat{R}(x, \xi, i\omega^2) \ U_{\hat{R}}(i\omega^2) \begin{bmatrix} w_0''(\xi) \\ v_0(\xi) \end{bmatrix} d\xi \ ,$$

where $\hat{R}(x, \xi, i\omega^2)$ is the 2×2 matrix kernel given by

$$\hat{R}(x, \xi, i\omega^2) = \hspace{6cm} (4.14)$$

$$\frac{1}{4}\left[\begin{array}{c}\omega^{-2}\left[\dfrac{\sinh \omega x}{\sinh \omega \pi} - \dfrac{\sin \omega x}{\sin \omega \pi}\right]\\[3mm] i\left[\dfrac{\sinh \omega x}{\sinh \omega \pi} - \dfrac{\sin \omega x}{\sin \omega \pi}\right]\end{array}\right]\left(-\left[\dfrac{\sinh \omega \xi}{\sinh \omega \pi} + \dfrac{\sin \omega \xi}{\sin \omega \pi}\right], \ i\left[\dfrac{\sinh \omega \xi}{\sinh \omega \pi} - \dfrac{\sin \omega \xi}{\sin \omega \pi}\right]\right).$$

Since the norm associated with (4.05) is just the $L^2[0,\pi] \times L^2[0,\pi]$ norm of $\left[\begin{array}{c}\tilde{w}\\v\end{array}\right]$, $\tilde{w} = w''$, in order to study the closed-loop resolvent we need only examine the operator $\delta R_{\tilde{K}}(i\omega^2)$ given by

$$\delta R_{\tilde{K}}(i\omega^2) \left[\begin{array}{c}\tilde{w}_0\\v_0\end{array}\right] = -\tilde{K} \int_0^\pi \tilde{R}(x,\xi,i\omega^2) \ U_{\tilde{K}}(i\omega^2) \left[\begin{array}{c}\tilde{w}_0(\xi)\\v_0(\xi)\end{array}\right] d\xi \ ,$$

where

$$\tilde{R}(x,\xi,i\omega^2) = \tag{4.15}$$

$$\frac{1}{4}\left[\begin{array}{c}\left[\dfrac{\sinh \omega x}{\sinh \omega \pi} + \dfrac{\sin \omega x}{\sin \omega \pi}\right]\\[3mm] i\left[\dfrac{\sinh \omega x}{\sinh \omega \pi} - \dfrac{\sin \omega x}{\sin \omega \pi}\right]\end{array}\right]\left(-\left[\dfrac{\sinh \omega \xi}{\sinh \omega \pi} + \dfrac{\sin \omega \xi}{\sin \omega \pi}\right], \ i\left[\dfrac{\sinh \omega \xi}{\sinh \omega \pi} - \dfrac{\sin \omega \xi}{\sin \omega \pi}\right]\right).$$

Let us note that the output feedback relation (cf.(1.02), (3.01)), applied to the present instance, becomes

$$u(t) = -\tilde{K} \frac{\partial^2 w}{\partial t \partial x}(\pi,t) + v(t) = -\tilde{K} \ y(t) + v(t) \ . \tag{4.16}$$

For $v(t) \equiv 0$ we have the homogeneous boundary condition

$$\frac{\partial^2 w}{\partial x^2}(\pi,t) + \tilde{K} \frac{\partial^2 w}{\partial t \partial x}(\pi,t) = 0 \ . \tag{4.17}$$

It is easy to see that the system consisting of (4.01) and (4.02), with the last condition of (4.02) replaced by (4.17), is dissipative with respect to the norm associated with the inner product (4.05) when $\tilde{K} > 0$. Applying the Lumer - Phillips theorem ([14]) we can see that the indicated feedback relation results in a closed-loop system

corresponding to a contraction semigroup. So, although neither the
input element b nor the output element (also b in this case) are ad-
missible, and even though they are likewise not jointly admissible,
for the open loop system ($\mathcal{K} = 0$), it is nevertheless true that $\mathcal{K}$
is a totally admissible gain coefficient (cf. §3) for $\mathcal{K} > 0$ (for
all values of $\mathcal{K}$, in fact).

As noted in §3, $T_{\mathcal{K}}(\lambda)$ is bounded in the closed half plane
Re $\lambda \geq 0$ just in case $U_{\mathcal{K}}(\lambda)$ is, and in the present case (cf.(3.08)),
with $\lambda = i\omega^2$,

$$U_{\mathcal{K}}(i\omega^2) = \frac{1}{1 + \mathcal{K}\,\dfrac{i\omega}{2}\left[\coth \omega\pi - \cot \omega\pi\right]} . \qquad (4.18)$$

The poles can be shown to lie in the first and third quadrants of
the ω plane. The second and fourth quadrants of the ω plane corres-
pond to the right half λ plane. On the real axis the boundedness of
$U_{\mathcal{K}}(\lambda)$ is evident because the term in parentheses is real. Replacing
ω by $i\omega$ and use of trigonometric identities establ.
sult on the imaginary axis. Carefully examining the functio...
restriction to the second (or fourth) quadrant of the lines Re $\omega = k$,
Im $w = k$ and using growth and periodicity properties along with the
maximum principle we can obtain boundedness in the rectangles

$$0 \leq \text{Im } \omega \leq k, \quad -k \leq \text{Re } \omega \leq 0$$

in the second quadrant and their counterparts in the fourth, the
bound being independent of k. The boundedness of $U_{\mathcal{K}}(\lambda)$, and hence
of $T_{\mathcal{K}}(\lambda)$, in the closed right half λ plane follows and we conclude
the admissibility of positive gain coefficients $\mathcal{K}$.

Now we ask if, for a particular gain $\mathcal{K}$, we can establish total
admissibility, i.e., the existence of a strongly continuous closed
loop semigroup, by the methods of §3, notwithstanding the non-admis-
sibility results cited above which prevent the use of Theorem 3.2 ?
Further, can we do this without the assumption that $\mathcal{K}$ is positive?
To answer this we define $R_{\mathcal{K}}(\lambda)$ as in §3 and note that we must exam-
the properties of the resolvent difference

$$\delta R_{\mathcal{K}}(\lambda) \equiv R_{\mathcal{K}}(\lambda) - R(\lambda) = \mathcal{K}\, U_{\mathcal{K}}(\lambda)\, R(\lambda)\, bb^{*}\, R(\lambda)$$

between $R_{\mathcal{K}}(\lambda)$ and the original resolvent operator. At issue is the conditional strong convergence and t - continuity of the integral

$$\frac{1}{2\pi i}\int_{\Gamma_{\alpha}} e^{\lambda t}\, \delta R_{\mathcal{K}}(\lambda)\, d\lambda \qquad (4.19)$$

for $\alpha > 0$ and $t > 0$ and the strong convergence of this integral to zero as $t \downarrow 0$, the latter equivalent to showing that

$$\lim_{t\downarrow 0} S_{\mathcal{K}}(t)x_0 = \lim_{t\downarrow 0}\frac{1}{2\pi i}\int_{\Gamma_{\alpha}} e^{\lambda t} R_{\mathcal{K}}(\lambda)\, d\lambda\, x_0 = x_0 \ , \qquad (4.20)$$

which can, in turn, be seen equivalent to the completeness of the eigenfunctions of the closed loop semigroup generator.

Since

$$\lim_{\omega \to \infty} \coth \omega\pi = 1 \ ,$$

and since an elementary analysis of the denominator in (4.18) shows that its zeros, which are the poles of the closed loop transfer function, are given to first order in $1/|\omega|$ as $|\omega| \to \infty$ by

$$\omega_k = k + \frac{1}{4} + \frac{4i}{\mathcal{K}(k+\frac{1}{4})} + \mathcal{O}\left(\frac{1}{k^2}\right), \quad k = 1,2,3,\ldots, \qquad (4.21)$$

corresponding to

$$\lambda_k = i\omega_k^2 = i\left(k+\frac{1}{4}\right)^2 - \frac{8i}{\mathcal{K}} + \mathcal{O}\left(\frac{1}{k}\right) \ ,$$

we suspect that the closed-loop semigroup may, in fact, be a group.

Replacing t by -t, the equation (4.01) is invariant while the closed-loop boundary condition (4.17) becomes

$$\frac{\partial^2 w}{\partial x^2}(\pi, t) - \Re \frac{\partial^2 w}{\partial t \partial x}(\pi, t) = 0 .$$

So we conclude that the semigroup is actually a group if we can establish the semigroup property for an arbitrary real $\Re$ rather than just for $\Re \geqq 0$ as must be assumed for application of the Lumer - Phillips theorem, as noted above.

Before getting into the proof proper, we require a definition and a lemma. A bounded sequence of vectors, $\Phi = \{\varphi_k\} \subset X$, a Hilbert space, will be said to the *uniformly ℓ^2- convergent* if the operator

$$T_\Phi : \ell^2 \to X$$

defined for $\xi = \{\xi_1, \xi_2, \xi_3, \dots\} \in \ell^2$ by

$$T_\Phi \xi = \sum_{k=1}^{\infty} \xi_k \varphi_k$$

is a bounded operator, so that, for some $B_\Phi > 0$,

$$\| T_\Phi \xi \|^2 \leqq B_\Phi \sum_{k=1}^{\infty} |\xi_k|^2 .$$

For later use we also note that, for fixed $x \in X$,

$$(x, T_\Phi \xi) = \sum_{k=1}^{\infty} (x, \varphi_k) \overline{\xi}_k$$

is a bounded linear functional on ℓ^2 and therefore $(x, \varphi_k) \in \ell^2$. Then we have the following lemma.

Lemma 4.1 *If Φ (as shown earlier) and $\Psi = \{\psi_k\}$ are uniformly ℓ^2-convergent sequences in X, then an operator series*

$$C = \sum_{k=1}^{\infty} c_k \, \psi_k \, \varphi_k^* \qquad\qquad (4.22)$$

with bounded c_k converges strongly to a bounded operator C on X. Further, if a sequence $\{C_\ell\}$ of such operators is defined with the corresponding sequences $\{c_{\ell k}\}$ uniformly bounded with $\lim\limits_{\ell \to 0} c_{\ell k} = 0$ for each k, then the sequence C_ℓ converges strongly to 0.

Proof. For the first part of the proof we may, w.l.o.g., replace the indicated coefficients c_k by 1, absorbing the coefficients in one of the sequences Φ or Ψ without changing the convergence property. We then need only note that the adjoint of T_Φ is the operator defined by

$$T_\Phi^* \, x \;=\; \{(x, \varphi_k)_X\} = \{ \, \eta_k \mid \eta_k = \varphi_k^* x, \; k = 1, 2, 3, \; \ldots \, \}$$

to see that the series is really $T_\Psi T_\Phi^*$ and hence represents a bounded operator since T_Ψ and T_Φ, hence T_Φ^*, are bounded operators. Let us denote the corresponding truncated operators by a superscript N, e.g.,

$$T_\Phi^N \, \xi \;=\; \sum_{k=1}^{N} \xi_k \, \varphi_k \; .$$

Then it is easy to see that

$$T_\Psi \, (T_\Phi^N)^* \;=\; T_\Psi^N \, T_\Phi^* \;=\; T_\Psi^N \, (T_\Phi^N)^* \; .$$

So, for $x \in X$,

$$T_\Psi \, T_\Phi^* \, x - \left[\sum_{k=1}^{N} \psi_k \, \varphi_k^* \right] x \;=\; \left[T_\Psi - T_\Psi^N \right] T_\Phi^* \, x$$

and the fact that

$$\lim_{N\to\infty} \left[T_{\Psi} - T_{\Psi}^{N} \right] T_{\phi}^{*} x = 0, \quad x \in X,$$

follows from the assumed ℓ^2 - convergence property of the sequence Ψ. Thus the series (4.22) converges strongly to $T_{\Psi} T_{\phi}^{*}$. To establish the second result we let x be a fixed vector in X and take $\epsilon > 0$. Since $(x, \varphi_k) \in \ell^2$ and the $c_{\ell k}$ are uniformly bounded, we can find K such that

$$\| \sum_{k=K}^{\infty} (x, \varphi_k) c_{\ell k} \Psi_k \| < \frac{\epsilon}{2},$$

uniformly for all ℓ. Then we need only find L such that for $\ell > L$

$$\| \sum_{k=1}^{K-1} (x, \varphi_k) c_{\ell k} \Psi_k \| < \frac{\epsilon}{2},$$

which follows from the boundedness of the (x, φ_k) and the assumed convergence to 0 of the $c_{\ell k}$ as $\ell \to \infty$, and we see that the proof is complete.

Having obtained this result, we return to the main question, that of establishing the convergence of the integral in (4.19). To this end we first replace Γ_{α} by

$$C_{\alpha} = \Gamma_{\alpha} \cup \Gamma_{-\alpha},$$

oriented positively, and then replace the two parallel lines of C_{α} by a sequence of closed rectangular contours C_{α}^{N}, $N = 1, 2, 3, \ldots$, where, for r_N to be described subsequently, C_{α}^{N} consists of the segments of C_{α} lying in $|\text{Im } \lambda| \leq r_N$ together with the connecting segments of $\text{Im } \lambda = \pm r_N$. Under the change of variable $\lambda = i\omega^2$ the straight lines $\Gamma_{\pm\alpha}$ are transformed into hyperbolae $(\omega = \rho + i\sigma)$

$$\sigma = \pm \frac{\alpha}{2\rho} \; ;$$

hence the rectangles C_α^N are transformed into contours K_α^N as shown in Figure 1.
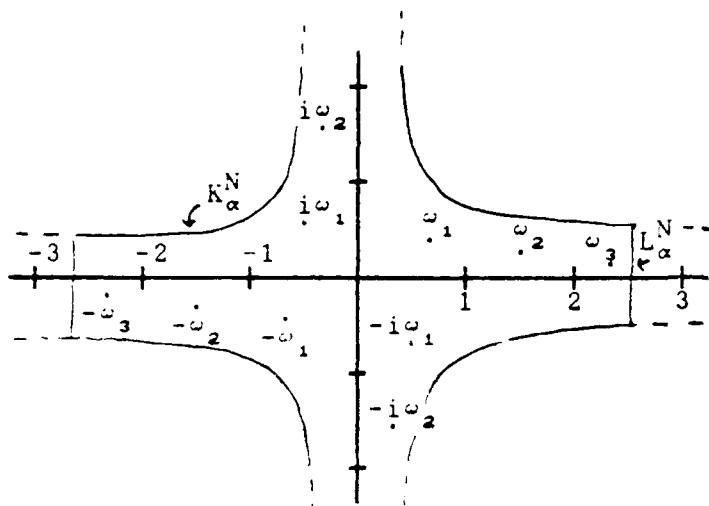

Figure 1.

The "end segments", L_α^N , are portions of the conjugate hyperbolae

$$\rho^2 - \sigma^2 = \pm r_N \; .$$

Traversing K_α^N once, C_α^N is traversed twice since ω is, of course, identified with $-\omega$ under $\lambda = i\omega^2$.

From the formula (4.15) we see that $\delta_R R(\lambda)$ is a 2 x 2 matrix of operators; the individual entries are much the same except for signs and their treatment, insofar as convergence of (4.19) is concerned, is virtually identical. We will treat only the "1,1" operator entry here, which is described by the kernel

$$\frac{\dfrac{\mathcal{R}}{4}\left[\dfrac{\sinh \omega\xi}{\sinh \omega\pi} + \dfrac{\sin \omega\xi}{\sin \omega\pi}\right] \cdot \left[\dfrac{\sinh \omega\xi}{\sinh \omega\pi} + \dfrac{\sin \omega\xi}{\sin \omega\pi}\right]}{1 + \dfrac{\mathcal{R}i\omega}{2}\left[\coth \omega\pi - \cot \omega\pi\right]} \tag{4.23}$$

acting via integration with respect to ξ on functions $\tilde{w}(\xi) \in L^2[0,\pi]$. Poles occur at the real and imaginary integers in the ω plane (actually, these poles of $\delta R_{\mathcal{R}}(\lambda)$ simply cancel, by subtraction, the corresponding poles of $R(\lambda)$, but we make no special use of that here) and at the zeros of the denominator in (4.23), which are the complex numbers $\pm \omega_k$, $\pm i\omega_k$, ω_k as in (4.21). It is clear that all of these poles will lie inside K_α if $\alpha > 0$ is chosen sufficiently large, whatever the sign of $\mathcal{R}$ may be. Since the k and ω_k are uniformly separated, r_N may be chosen so that K_α^N includes k, $-k$, ω_k, $-\omega_k$, ik, $-ik$, $i\omega_k$, $-i\omega_k$, $k \leq N$ (for sufficiently large k, at least), and so that K_α^N remains uniformly bounded away from all of these points for all N.

We carry out the multiplication indicated in the numerator of (4.23) and multiply numerator and denominator by $\sin \omega\pi \sinh \omega\pi$ to obtain

$$\frac{\dfrac{\mathcal{R}}{4}\left[\left[\dfrac{\sinh \omega x \sinh \omega\xi}{\sinh \omega\pi}\right]\sin \omega\pi + \sinh \omega x \sin \omega\xi + \sin \omega x \sinh \omega\xi + \left[\dfrac{\sin \omega x \sin \omega\xi}{\sin \omega\pi}\right]\sinh \omega\pi\right]}{\sin \omega\pi \sinh \omega\pi + \dfrac{\mathcal{R}i\omega}{2}\left[\cosh \omega\pi \sin \omega\pi - \sinh \omega\pi \cos \omega\pi\right]} \tag{4.24}$$

Let us first consider the "$(\sin \omega x \sin \omega\xi)/\sin \omega\pi$" term. The expression is used in the partial integral obtained from (4.19), with Γ_α replaced by C_α, by the change of variable $\lambda = i\omega^2$ ($d\lambda = 2i\omega\, d\omega$),

$$\frac{1}{2\pi i} \int_{K_\alpha^N} \frac{e^{i\omega^2 t} \dfrac{\sin \omega x \sin \omega \xi}{\sin \omega \pi} \, \omega \sinh \omega \pi \, d\omega}{\omega \left[\cosh \omega \pi \sin \omega \pi - \sinh \omega \pi \cos \omega \pi\right] - (2i/\Re) \sinh \omega \pi \sin \omega \pi} .$$

(4.25)

Computation of the residues at the real integers $\omega = k, -k$, yields

$$-\frac{2}{\pi} \, e^{ik^2 t} \sin kx \sin k\xi .$$

(4.25)

The "$(\sinh \omega x \sinh \omega \xi)/\sinh \omega \pi$" term will have similar residues at the points $\pm ik$.

The other poles of the integrand in (4.25) come from the zeros $\pm \omega_k, \pm i\omega_k$ of the denominator. Again treating only the $\sin \omega_k x \sin \omega_k \xi$ term, we see that the residue has the form

$$\frac{\omega_k \sinh \omega_k \pi \, e^{i\omega_k^2 t} \sin \omega_k x \sin \omega_k \xi}{\sin \omega_k \pi \, D'(\omega_k)} ,$$

(4.27)

where $D(\omega)$ denotes the denominator just referred to. An easy computation, using (4.21), shows that as $k \to \infty$

$$\frac{\omega_k \sinh \omega_k \pi}{\sin \omega_k \pi \, D'(\omega_k)} \to \frac{1}{\pi} ,$$

and thus (4.27) has the form

$$d_k \, e^{i\omega_k^2 t} \sin \omega_k x \sin \omega_k \xi , \quad d_k \to \frac{1}{\pi} .$$

(4.28)

The other terms in (4.24) are treated in much the same way except that wherever $\sinh \omega_k x$ or $\sinh \omega_k \xi$ occurs the expression is rearranged so that the term actually used is $\sinh \omega_k x / \sinh \omega_k \pi$, etc. This results in residues at the points ω_k with the forms

$$a_k \ e^{i\omega_k^2 t} \ \frac{\sinh \omega_k x \ \sinh \omega_k \xi}{\sinh^2 \omega_k \pi} \ , \quad b_k \ e^{i\omega_k^2 t} \ \frac{\sinh \omega_k x \ \sin \omega_k \xi}{\sinh \omega_k \pi} \ ,$$

$$c_k \ e^{i\omega_k^2 t} \ \frac{\sin \omega_k x \ \sinh \omega_k \xi}{\sinh \omega_k \pi} \ ,$$

(4.29)

with each of the coefficients a_k, b_k, c_k tending to $1/\pi$ as $k \to \infty$. An entirely similar situation occurs at the points $-\omega_k$, $\pm i\omega_k$, except that the roles of the sin and sinh functions are reversed at the pints $\pm i\omega_k$ as compared to $\pm\omega_k$. Noting that

$$\sin \omega_k \xi = \overline{\sin \overline{\omega_k} \xi}$$

and treating sinh $\omega_k \xi$ similarly, we may invoke Lemma 4.1 repeatedly to obtain the strong convergence of the operator series corresponding to each of the expressions shown in (4.28), noting that the appropriate uniform ℓ^2 - convergence property of the sin kx functions is well known, for the sin $\omega_k x$ (or sin $\overline{\omega_k}\xi$) may be obtained from [7], and for the functions sinh $\omega_k x$/sinh $\omega_k \pi$ (or sinh $\overline{\omega_k}\xi$/sinh $\overline{\omega_k}\pi$) may be obtained by the methods developed in [17]. From this we conclude, therefore, that the integrals over K_α^N, and hence over C_α^N, converge strongly as $N \to \infty$. The operator integrand (4.23) can be seen to be uniformly bounded with respect to N on the end segments L_α^N of K_α^N and therefore, since the lengths of these segments tend to zero as $N \to \infty$, the integral over K_α, equivalently over C_α, is strongly convergent, provided the integral over the infinite contour is approximated via integrals over the contours obtained from K_α^N or C_α^N by deleting the end segments. This restriction has to be imposed because the operator integrand (4.23) does not tend to zero, nor, for that matter, even remain bounded, as we move out to ∞ along the contours K_α, due to the path of integration moving ever closer to the poles at the points $\pm k$, $\pm ik$, $\pm\omega_k$, $\pm i\omega_k$. The convergence is quite strictly conditional on the precise sense of convergence we have indicated

here in our choice of location of the L_α^N.

Once we have the strong convergence of the integral over C_α, as explained in the preceding paragraph, the integral over $\Gamma_{-\alpha}$ can be shown to be equal to zero for $t > 0$ in the same way already developed in §3. The semigroup property also follows in the same way as we indicated there. Strong continuity for $t \geq 0$ of the integral in (4.20) follows from uniform continuity of $e^{\lambda t}$ with respect to t for λ in any compact set, the uniform boundedness of this function for bounded t and λ in a fixed vertical strip in the complex plane, and application of the second result of Lemma 4.1 to the operator series obtained from the integral in (4.20) with $e^{\lambda t}$ replaced by the difference $e^{\lambda(t+\delta t)} - e^{\lambda t}$.

The putative closed loop semigroup now has all of the desired properties save one. It is conceivable that the $S_\kappa(t)$ might just be the image of the actual closed loop semigroup under the action of a projection operator P which commutes with A_κ . To show that any such P must, in fact, be the identity operator, it suffices to show that (4.20) holds for any x_0 (i.e., w_0, v_0 in the present case) in the system state space. It is here that we use the boundedness of $U_\kappa(\lambda)$, equivalently that of $T_\kappa(\lambda)$, in a very essential manner, to establish strong continuity via an argument along the lines laid down in Theorem 3.4.

Let us consider, first of all, the homogeneous open loop system, (4.01), (4.02) with $u(t) \equiv 0$, and solutions $w(x,t)$ corresponding to certain initial states. Specifically, we consider initial states

$$w(x,0) = w_0(x), \quad \frac{\partial w}{\partial t}(x,0) = v_0(x), \qquad (4.30)$$

with $(w_0,v_0) \in D(A^2)$, so that $w_0 \in H^8[0,\pi]$, $v_0 \in H^6[0,\pi]$ and the boundary conditions

$$w_0(0) = w_0''(0) = w_0^{(iv)}(0) = w_0^{(vi)}(0) = 0 , \qquad (4.31)$$

$$w_0(\pi) = w_0^{''}(\pi) = w_0^{(iv)}(\pi) = w_0^{(vi)}(\pi) = 0 , \qquad (4.32)$$

$$v_0(0) = v_0^{(iv)}(0) = 0 , \qquad (4.33)$$

$$v_0(\pi) = v_0^{(iv)}(\pi) = 0 , \qquad (4.34)$$

are all satisfied. Then we impose the further conditions

$$w_0^{(v)}(\pi) = 0, \quad v_0^{'}(\pi) = 0 . \qquad (4.35)$$

Since the state space is (4.04), it is readily seen that the set $\mathbb{D}$ of all initial states (w_0, v_0) just described is dense in that space.

Since such an initial state lies in $\mathbb{D}(A^2)$, the resulting solution $w(x,t)$ has this property for all $t \geq 0$, i.e.,

$$w(\cdot, t) \in H^8[0, \pi], \quad \frac{\partial w}{\partial t}(\cdot, t) \in H^6[0, \pi] ,$$

and, in addition, boundary conditions parallelling (4.31) - (4.34) are satisfied. Since the system is conservative and the norm used is equivalent to system energy, we know that

$$\left\| A^2 \begin{bmatrix} w(\cdot; t) \\ v(\cdot; t) \end{bmatrix} \right\| = \left\| A^2 \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \right\| , \quad t \geq 0;$$

in particular the norm on the left is uniformly bounded. The norm used here is, of course, the one associated with the inner product (4.05). Moreover, we know that $A^2 \begin{bmatrix} w(\cdot; t) \\ v(\cdot; t) \end{bmatrix}$ is continuous with respect to t, $t \geq 0$, in the state space (4.04). Since point evaluation of the $(n-1)$-st derivative is continuous on $H^n(I)$, where I is any closed interval of R^1, the output from the solution $w(x,t)$ via (4.03), $y_0(t)$, is such that its second t-derivative,

$$\frac{d^2 y_0}{dt^2}(t) = \frac{\partial^3 v}{\partial t^2 \partial x}(\pi, t) = \frac{\partial^5 v}{\partial x^5}(\pi, t)$$

is continuous and uniformly bounded for $t \geq 0$. Further, we can see

from (4.35) and (4.01) that

$$y_0(0) = \frac{\partial^2 w}{\partial t \partial x}(\pi,0) = v_0'(\pi) = 0,$$

$$y_0'(0) = \frac{\partial^3 w}{\partial t^2 \partial x}(\pi,0) = -\frac{\partial^5 w}{\partial x^5}(\pi,0) = -w_0^{(v)}(\pi) = 0.$$

From this it follows that $\hat{y}_0(\lambda)$, the Laplace transform of $y_0(t)$, is such that $\lambda^2 \hat{y}_0(\lambda) \in H^2(\mathbb{C}_a)$ for any $a > 0$. But we have seen earlier that if we take a_κ sufficiently large, so that it lies to the right of any poles of $U_\kappa(\lambda)$ in the right half plane (these, of course, occur only for $\kappa < 0$), then $U_\kappa(\lambda)$ is uniformly bounded for Re $\lambda \geq a_\kappa$. Therefore, if we set

$$\hat{y}_\kappa(\lambda) = U_\kappa(\lambda)\,\hat{y}_0(\lambda),$$

we shall also have $\lambda^2 \hat{y}_\kappa(\lambda) \in H^2(\mathbb{C}_{a_\kappa})$. Letting $u_\kappa(t)$ denote the

inverse Laplace transform of $\hat{u}_\kappa(\lambda)$, we clearly have

$$u_\kappa(0) = u_\kappa'(0) = 0 \qquad\qquad (4.36)$$

and $u_\kappa(t)$, $u_\kappa'(t)$ and $u_\kappa''(t)$ all lie in $L^2_{a_\kappa}[0,\infty)$.

Now, following a procedure originally introduced by Fattorini ([9]), we make the transformation

$$w_\kappa(x,t) = z_\kappa(x,t) + \zeta(x)u_\kappa(t), \qquad\qquad (4.37)$$

wherein $w_\kappa(x,t)$ is to be a solution of the system (4.01), (4.02) with $u(t)$ replaced by $u_\kappa(t)$ and $\zeta(x)$ is the solution of

$$\frac{d^4\zeta}{dx^4}(x) = 0$$

satisfying the boundary conditions

$$\zeta(0) = \zeta''(0) = \zeta(\pi) = 0, \quad \zeta''(\pi) = 1 \ ,$$

i.e., $\zeta(x) = -\frac{\pi}{6}x + \frac{x^3}{6\pi}$. We find then that $z_R(x,t)$ is a solution of

$$\frac{\partial^2 z_R}{\partial t^2} + \frac{\partial^4 z_R}{\partial x^4} + \zeta(x)u_R''(t) \qquad (4.38)$$

satisfying the homogeneous boundary conditions

$$z_R(0,t) = 0 \ , \quad \frac{\partial^2 z_R}{\partial x^2}(0,t) = 0 \ , \quad z_R(\pi,t) = 0, \quad \frac{\partial^2 z_R}{\partial x^2}(\pi,t) = 0. \quad (4.39)$$

The solution $w_R(x,t)$ is to satisfy the same initial conditions as $w(x,t)$ discussed above, i.e., (4.30) through (4.35), so we shall have

$$z_R(x,0) = w_0(x) \ , \quad \frac{\partial z_R}{\partial t}(x,0) = v_0(x) \ , \qquad (4.40)$$

since u_R satisfies (4.36).

Now the system (4.38), (4.39) is of the form

$$\dot{z}_R = A \, x_R + g \, v$$

with

$$g = \begin{bmatrix} 0 \\ \zeta \end{bmatrix} \in X \ ,$$

X denoting the state space (4.04) here, and $v = u_R''$ locally square integrable. For such a system, with initial state (4.40) in X, it is well known (see [11] e.g.) that the resulting solution z_R exists, is unique and is continuous in X, assuming, also continu-

ously as $t \downarrow 0$, the given initial state. It follows that (4.37) provides a solution $w_{\hat{\kappa}}(x,t)$ of (4.01), (4.02) with the same properties, $u(t)$ replaced by $u_{\hat{\kappa}}(t)$. The rapid decay properties of $u_{\hat{\kappa}}(\lambda)$ as Im $\lambda \to \infty$ allow one to see that the transform of $w_{\hat{\kappa}}(x,t)$, $v_{\hat{\kappa}}(x,t)$ is

$$\begin{bmatrix} \hat{w}_{\hat{\kappa}}(x,\lambda) \\ \hat{v}_{\hat{\kappa}}(x,\lambda) \end{bmatrix} = R(\lambda) \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} + R(\lambda) \, b \, \hat{u}_{\hat{\kappa}}(\lambda) \qquad (4.41)$$

and that the inhomogeneous part of $w_{\hat{\kappa}}(x,t)$ can be recovered from the second term via an inversion integral of the form (2.06). But it is equally true that

$$\hat{u}_{\hat{\kappa}}(\lambda) = \hat{\kappa} \, \hat{y}_{\hat{\kappa}}(\lambda) = \hat{\kappa} \, U_{\hat{\kappa}}(\lambda) \, \hat{y}_0(\lambda) = \hat{\kappa} \, U_{\hat{\kappa}}(\lambda) \, b^* \, R(\lambda) \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \quad (4.42)$$

and, substituting (4.42) into (4.41) and comparing with the form of $R_{\hat{\kappa}}(\lambda)$ shown earlier, we conclude that it is also true that

$$\begin{bmatrix} w_{\hat{\kappa}}(x,t) \\ v_{\hat{\kappa}}(x,t) \end{bmatrix} = S_{\hat{\kappa}}(t) \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} .$$

What we have shown, then, is that there is a set of initial states $x_0 = (w_0, v_0)$, dense in the state space, for which (4.20) is true. Since we have shown the operators $S_{\hat{\kappa}}(t)$ to be uniformly bounded on any bounded t - interval, we conclude that (4.20) must, in fact, be true for all initial states in the state space (4.04). We have, therefore, proved

Theorem 4.2 *The closed loop system consisting of the equation* (4.01), *the first three equations of* (4.02), *and the boundary condition* (4.17), *corresponds to a strongly continuous group of operators* $S_{\hat{\kappa}}(t)$ *on the state space* (4.04) *for every real value of* $\hat{\kappa}$.

The arguments given here also provide, in essence, the omitted proof of Theorem 3.4.

5. Approximation of Input - Output Operators.

No analysis of transfer function methods for distributed parameter systems can be considered complete if it does not address the question of approximation. In other words, if we have a sequence of finite dimensional systems

$$\dot{x}_n = A_n \, x_n + b_n \, u \, , \qquad (5.01)$$

$$y_n = c_n^* x_n \, , \quad n = 1,2,3, \, \ldots \, , \qquad (5.02)$$

approximating (1,01),(1.02 in some appropriate sense (in particular, we would expect the sequence of approximating generators A_n to obey the Trotter - Kato conditions, as set forth in [12],[21], and if we define

$$T_n(\lambda) = c_n^* \Big[\lambda I_n - A_n \Big]^{-1} b_n \, ,$$

as the corresponding approximating transfer function, then what can we say about the convergence of the $T_n(\lambda)$ to $T(\lambda)$, and hence the convergence of the approximating input-output operators $J_{n,\rho}$ to the input-output operator (1.10) associated with the complete system (1.01),(1.02):

$$J_\rho : \ L^2[0,\infty) \ \rightarrow \ L_\rho^2[0,\infty), \quad \rho \geqq \gamma = \max \, \{a,0\} \ ?$$

Here a is defined with reference to A as in §1 and, for any $\rho \geqq 0$, $L_\rho^2[0,\infty)$ is defined as in (1.11). The reader will recall that total admissibility of the input - output pair b,c implies that the operator J_ρ, as defined here, is a bounded operator for some $\rho > a$, where a is as defined in Theorem 1.1.

In this paper we consider primarily modal (eigenfunction) approximations, corresponding to replacement of the standard series representation of $T(\lambda)$,

$$T(\lambda) = \sum_{k=1}^{\infty} \frac{\overline{c}_k b_k}{\lambda - \lambda_k}$$

(assuming $T(\lambda)$ has distinct eigenvalues λ_k and an associated Riesz
basis of eigenvectors $\left\{ \varphi_k \right\}$ as discussed in §3 above), by increasing
partial sums $T_n(\lambda)$ (by increasing we mean that all terms in $T_n(\lambda)$
are strictly incl·ded in the next partial sum $T_{n+1}(\lambda)$, $n = 1,2,3,\ldots$)
converging uniformly to $T(\lambda)$ in compact subsets of Re $\lambda \geq \rho$. Asso-
ociating with each $T_n(\lambda)$ the corresponding input - output operator
$J_{n,\rho} : L^2[0,\infty) \rightarrow L_\rho^2[0,\infty)$, we will say that the sequence $\{T_n\}$ $(\{J_{n,\rho}\})$
(weakly, strongly, uniformly) approximates T (J_ρ) according as the
operator sequences converge to the indicated limit in the (weak,
strong, uniform) operator topology. The basic result is

Theorem 5.1 *Let b and c be admissible input and output elements for
the system* (1.01),(1.02) *which are also jointly admissible, i.e., b
and c are totally admissible for* (1.01),(1.02). *Then* $\{T_n\}$ $(\{J_{n,\rho}\})$,
as described above, strongly approximate T (J_ρ), *for a given* $\rho \geq 0$,
if there is a positive number τ_ρ *such that*

$$|T_n(\lambda)| \leq \tau_\rho , \quad Re \lambda \geq \rho , \quad n = 1,2,3,\ldots . \tag{5.03}$$

(Of course the same bound will then be satisfied by $T(\lambda)$.*)*

Further, if $\rho = 0$ *and* (5.03) *is not true, or if for* $\rho > 0$ *there is
a* $\delta > 0$ *and points* σ_n *such that*

$$\lim_{n\to\infty} \inf_{|\sigma-\sigma_n|\leq\delta} \{|T_n(\rho+i\sigma)|\} = \infty \tag{5.04}$$

*then the indicated strong approximation result definitely does not
hold.*

Remark. It seems likely that (5.03) is necessary for the strong ap-
proximation whatever the value of $\rho > 0$ but, since the present result

is adequate for our present purposes, we do not pursue such a theorem here. If the b_k and c_k are uniformly bounded, $\rho > 0$, and the λ_k have a maximum density as described in §3 then the failure of (5.03) immediately gives (5.04) because the derivatives $T_n{}'(\lambda)$ may then be seen to be uniformly bounded (even though the $T_n(\lambda)$ are not). These results are reminiscent of and, of course, related to the Trotter - Kato ([21]) and Lax - Richtmeyer ([13]) results.

Proof. If the $T_n(\lambda)$ satisfy uniform bounds (5.03) and converge uniformly to $T(\lambda)$ on compact sets, it is quite simple to see that, for any $\varphi \in H^2(\mathbb{C}_\rho^+)$, the Hardy space on the half plane Re $\lambda \geq \rho$, the sequence $\{T_n(\lambda)\varphi(\lambda)\}$ converges to $T(\lambda)\varphi(\lambda)$ relative to the norm in that space. This yields the strong convergence of the T_n to T, which, in turn, also gives the strong convergence of the operators $J_{n,\rho}$ to J_ρ since the Plancherel theorem implies that strong convergence of the $T_n(\lambda)$, interpreted as multiplication operators from $H^2(\mathbb{C}^+)$ to $H^2(\mathbb{C}_\rho^+)$, to $T(\lambda)$ is equivalent to strong convergence of the corresponding convolution operators $J_{n,\rho}$ to J_ρ. So the sufficiency part of the theorem is proved.

The necessity part for $\rho = 0$ is quite direct. If the indicated uniform boundedness does not hold, the maximum principle shows that it cannot hold on Re $\lambda = 0$. Then by looking at functions

$$\varphi_{r,\sigma}(\lambda) = \frac{r}{\lambda+r-i\sigma}, \quad r \downarrow 0, \quad (5.05)$$

one can readily see that the norm of $T_n(\lambda)$ as a multiplication operator is equal to the maximum of its modulus on Re $\lambda = 0$ and hence tends to ∞ as $n \to \infty$. The Principle of Uniform Boundedness then shows that the $T_n(\lambda)$ cannot converge to $T(\lambda)$ in the strong topology of bounded operators on $H^2(\mathbb{C}^+)$. The Plancherel theorem immediately extends this negative result to the input output operators $J_{n,\rho}$, J_ρ.

The proof for $\rho > 0$ is much the same except that the functions (5.05) cannot be used as shown. Instead we fix $r > 0$ and for a given σ we form

$$\varphi_{r,\sigma}(\lambda) = \frac{r+\rho}{\lambda+r-i\sigma} \qquad (5.06)$$

which has the value 1 at $\lambda = \rho+i\sigma$ and has modulus < 1 everywhere else on the line Re $\lambda = \rho$. Then we find $N = N(\delta)$ such that the function

$$\psi_{r,\sigma}(\lambda) = (\varphi_{r,\sigma}(\lambda))^N$$

satisfies

$$\int_{\sigma-\delta}^{\sigma+\delta} |\psi_{r,\sigma}(\rho+i\tau)|^2 \, d\tau \;\geq\; \tfrac{1}{2} \, \|\psi_{r,\sigma}\|^2_{L^2(\Gamma_\rho)} \; .$$

Then the ψ_{r,σ_n} are uniformly bounded in $H^2(\mathbb{C}^+)$ but it is quite direct to see that the functions $T_n(\lambda) \, \psi_{r,\sigma_n}(\lambda)$ have norms tending to

infinity in $H^2(\mathbb{C}_\rho^+)$. The negative result then follows as in the previous proof for $\rho = 0$ and the theorem is proved.

We proceed now to show that the strong approximation property of the operators $J_{n,\rho}$ is, in fact, a rather delicate business, depending on very particular properties of the system being studied.

We consider a uniform stretched string of length π with endpoint fixed at the end $x = 0$ and a free endpoint, subject to a laterally directed force at the end $x = \pi$. The complete system may be taken to be

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = 0 \; , \qquad (5.07)$$

$$w(0,t) = 0, \quad \frac{\partial w}{\partial x}(\pi,t) = u(t) \; . \qquad (5.08)$$

Further, we define an output

$$y(t) = \pi \frac{\partial v}{\partial t}(\pi, t) , \qquad (5.09)$$

and we study the input - output operator

$$J_\rho: L^2[0,\infty) \rightarrow L^2[0,\infty) , \quad \rho > 0 .$$

In this case we have the transfer function relation

$$\hat{y}(\lambda) = T(\lambda) \hat{u}(\lambda) ,$$

an exercise similar to, but even simpler than, that in §4 showing that

$$T(\lambda) = \pi \tanh(\pi\lambda) = \sum_{k=1}^{\infty} \left[\frac{1}{\lambda - (k-\frac{1}{2})i} + \frac{1}{\lambda + (k-\frac{1}{2})i} \right] , \quad (5.10)$$

the series, as grouped here, being uniformly convergent in compact subsets of either the right or left half plane. It is an elementary exercise (see [11] for details) to check that the input element and the output element are admissible here. Joint admissibility is a direct application of the material in §2.

If we use a modal approximation to the system (5.07), (5.08), (5.09) based on the first n natural modes of vibration we obtain a system whose transfer function is

$$T_n(\lambda) = \sum_{k=1}^{n} \left[\frac{1}{\lambda - (k-\frac{1}{2})i} + \frac{1}{\lambda + (k-\frac{1}{2})i} \right] .$$

Each of these transfer functions is bounded for Re $\lambda \geq \rho > 0$ but, as we will see, they are not uniformly bounded and, by the argument given above, the associated input-output operators $J_{n,\rho}$ cannot converge strongly to the operator J_ρ corresponding to the transfer function $T(\lambda)$. To see that this is the case, we let μ be a complex number with Re $\mu \geq \rho$ and we consider the values

$$T_n((n-\frac{1}{2})i + \mu) , \quad n = 1,2,3, \ldots .$$

We have

$$T_n((n-\tfrac{1}{2})i + \mu) = \sum_{k=1}^{n} \left[\frac{1}{\mu + (n-k)i} + \frac{1}{\mu + (n+k-1)i} \right] =$$

$$= \sum_{j=0}^{2n-1} \frac{1}{\mu + ji} = i\left[\int_0^2 \frac{dx}{\mu i - k} + E(\mu i,n) \right] =$$

$$i\left[\log(\mu i) - \log(\mu i-2n) + E(\mu i,n) \right] ,$$

where $E(\mu i,n)$ is the familiar error term incurred in replacing the sum by the integral. Since that error term is well known ([23]) to be uniformly bounded for all n and for μ as described, we conclude that

$$\lim_{n\to\infty} |\ T_n((n-\tfrac{1}{2})i + \mu)\ | = \infty$$

for any such μ. Since the λ_k satisfy the bounded density property (quite trivially) in this case the negative result as regards strong approximation of T (J_ρ) by the T_n $(J_{n,\rho})$ follows from the remark made following the statement of Theorem 5.1.

The conclusion, that the operators $J_{n,\rho}$ do not converge strongly to the input - output operator J_ρ associated with the infinite dimensional process, must be regarded as implying a serious limitation of the modal approximation process as it applies to this energy conserving system. It will be straightforward for the reader to verify that this situation is unchanged if the equation (5.07) is replaced by

$$\frac{\partial^2 w}{\partial t^2} + 2\gamma \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = 0 ,$$

representing a vibrational or wave propagation system with so-called "viscous" damping.

Now, taking

$$A = -\frac{\partial^2}{\partial x^2},$$

with $\mathcal{D}(A)$ determined by homogeneous boundary conditions compatible with (5.08), we consider the system with "square root" structural damping

$$w'' + 2\gamma A^{1/2} w' + A w = \delta_\pi u(t), \quad \gamma > 0, \qquad (5.11)$$

or, in first order form with $v = w'$,

$$\frac{d}{dt}\begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A & -2\gamma A^{1/2} \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \delta_\pi \end{bmatrix} u(t),$$

where A is the positive square root of the positive self-adjoint operator A. This corresponds to the admissible input element (see [11], e.g.)

$$b = \begin{bmatrix} 0 \\ \delta_\pi \end{bmatrix},$$

which is just another way of describing the input process indicated by the second equation in (5.08). The output process is again (5.09) which corresponds to the admissible output element

$$c^* = b^* = \begin{bmatrix} 0 & \delta_\pi \end{bmatrix}$$

just as in the case of the beam example of §4 except that δ_π' there is replaced by δ_π here.

After some calculation, which need not be exhibited here, it may be seen that, with

$$\omega_k \equiv k - \frac{1}{2}, \quad k = 1,2,3, \ldots,$$

the transfer function for this system is

$$T_\gamma(\lambda) = \sum_{k=1}^\infty \frac{\lambda}{\lambda^2 - 2\,\omega_k\,\lambda\,\cos\psi + \omega_k^2} =$$

$$= \sum_{k=1}^\infty \frac{1}{2\sinh i\psi} \left[\frac{e^{i\psi}}{\lambda - \omega_k e^{i\psi}} - \frac{e^{-i\psi}}{\lambda - \omega_k e^{-i\psi}} \right] , \qquad (5.12)$$

where ψ is the angle (between $\pi/2$ and π) such that

$$-\cos\psi = \gamma .$$

The transfer functions corresponding to modal approximation are then

$$T_{\gamma,n}(\lambda) = \sum_{k=1}^n \frac{1}{2\sinh i\psi} \left[\frac{e^{i\psi}}{\lambda - \omega_k e^{i\psi}} - \frac{e^{-i\psi}}{\lambda - \omega_k e^{-i\psi}} \right] .$$

It is again easy to verify uniform convergence of the $T_{\gamma,n}(\lambda)$ to $T_\gamma(\lambda)$, as $n \to \infty$, in compact subsets of Re $\lambda \gtrsim 0$. But now we can show more; we can show that the $T_{\gamma,n}(\lambda)$ are also uniformly bounded for Re $\lambda \gtrsim 0$, so that, as multiplication operators from $H^2(\mathbb{C}^+)$ to $H^2(\mathbb{C}_\rho^+)$, $\rho \gtrsim 0$, the $T_{\gamma,n}(\lambda)$ do converge strongly to $T_\gamma(\lambda)$, with the Plancherel theorem then implying the same conclusion with regard to the corresponding input-output operators $J_{\gamma,n,\rho}$, $J_{\gamma,\rho}$ from $L^2[0,\infty)$ to $L_\rho^2[0,\infty)$. To establish the indicated boundedness property it is clearly enough to study the functions

$$T_n(\lambda) = \sum_{k=1}^n \left[\frac{e^{i\psi}}{\lambda - \omega_k e^{i\psi}} - \frac{e^{-i\psi}}{\lambda - \omega_k e^{-i\psi}} \right] .$$

To study the sum of the first terms indicated, we let $\mu = e^{-i\psi}\lambda$ and observe that this sum becomes

$$\sum_{k=1}^{n} \frac{1}{\mu - (k-\frac{1}{2})} = \int_{\frac{1}{2}}^{n+\frac{1}{2}} \frac{dx}{\mu - x} + E(\mu,n) \, ,$$

where $E(\mu,n)$, the error term, may be seen to be uniformly bounded for Re $\lambda \geq 0$ and all positive integers n. Clearly

$$\int_{\frac{1}{2}}^{n+\frac{1}{2}} \frac{dx}{\mu - x} = \log (x - \mu) \Big|_{\frac{1}{2}}^{n+\frac{1}{2}} = \log ((n+\tfrac{1}{2}) - \mu) - \log (\tfrac{1}{2} - \mu)$$

$$= \log ((n+\tfrac{1}{2})e^{i\psi} - \lambda) - \log (\tfrac{1}{2}e^{i\psi} - \lambda) \, .$$

Similarly, the sum of the second terms reduces to a comparable error term plus

$$\log ((n+\tfrac{1}{2})e^{-i\psi} - \lambda) - \log (\tfrac{1}{2}e^{-i\psi} - \lambda) \, .$$

Subtracting the second from the first we see that

$$\mathbf{I}_n(\lambda) = \log \left[\frac{(n+\tfrac{1}{2})e^{i\psi} - \lambda}{(n+\tfrac{1}{2})e^{-i\psi} - \lambda} \right] - \log \left[\frac{\tfrac{1}{2}e^{i\psi} - \lambda}{\tfrac{1}{2}e^{-i\psi} - \lambda} \right]$$

$$+ E(e^{-i\psi}\lambda,n) - E(e^{i\psi}\lambda,n) \, .$$

The last three terms are uniformly bounded for all n and Re $\lambda \geq 0$; we need only consider the first. Since the argument of the quotient is bounded, it is only necessary to show that

$$\left| \frac{(n+\tfrac{1}{2})e^{i\psi} - \lambda}{(n+\tfrac{1}{2})e^{-i\psi} - \lambda} \right| \tag{5.13}$$

is bounded and bounded away from zero uniformly for positive integers n and Re $\lambda \geq 0$. For Im $\lambda \geq 0$ this magnitude is clearly ≤ 1 so we only need to show that it is bounded away from zero. A very easy geometric argument shows that it is enough to do this for Re $\lambda = 0$. So,

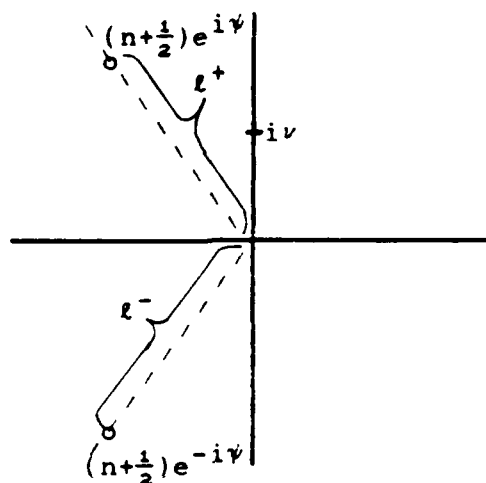referring to Figure 2, we let $\lambda = i\nu$, $\nu \gtreqless 0$,

Figure 2.

and we consider the ratio, reciprocal to (5.13),

$$Q_n(\nu) = \ell^-(n,\nu)/\ell^+(n,\nu) \ . \tag{5.14}$$

This is clearly a continuous function of ν, the value at $\nu = 0$ is 1 and

$$\lim_{\nu \to \infty} Q_n(\nu) = 1.$$

So the maximum of the ratio (5.14) occurs at some finite $\nu_n > 0$, since there clearly are ν for which the ratio exceeds 1. It can be seen that

$$\nu_n > (n+\tfrac{1}{2})\sin \psi$$

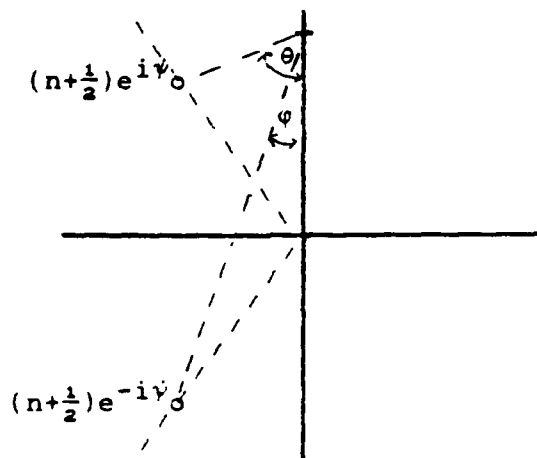and, in fact, occurs at the point where, now referring to Figure 3,

Figure 3.

$$\varphi + \frac{\pi}{2} + \theta = \pi \ , \quad \text{i.e.,} \quad \varphi + \theta = \frac{\pi}{2} \ .$$

Since for r, R > 0

$$\left| \frac{r\, e^{-i\hat{\varphi}} - i\nu}{r\, e^{i\hat{\varphi}} - i\nu} \right| = \left| \frac{R\, e^{-i\hat{\varphi}} - i(R/r)\nu}{R\, e^{i\hat{\varphi}} - i(R/r)\nu} \right| \equiv \left| \frac{R\, e^{-i\hat{\varphi}} - i\hat{\nu}}{R\, e^{i\hat{\varphi}} - i\hat{\nu}} \right| \ ,$$

the maximum value of $Q_n(\nu)$ (though not the point where it is assumed) is independent of n and we conclude, therefore, that the approximating transfer functions $T_{\gamma,n}(\lambda)$ are uniformly bounded for all positive integers K and $\mathrm{Re}\ \lambda \gtreqless 0$ and $\mathrm{Im}\ \lambda \gtreqless 0$. An entirely similar argument establishes the same result For $\mathrm{Re}\ \lambda \gtreqless 0$ and $\mathrm{Im}\ \lambda \lesseqgtr 0$. Combining this uniform boundedness with the uniform convergence of the $T_{\gamma,n}(\lambda)$ to $T_\gamma(\lambda)$ it is elementary to show that the $T_{\gamma,n}(\lambda)$ converge strongly to $T_\gamma(\lambda)$ as multiplication operators from $H^2(\mathbb{C}^+)$ to $H^2(\mathbb{C}_\rho^+)$ for $\rho \geq 0$. The corresponding result for the $J_{\gamma,n,\rho}$ and $J_{\gamma,\rho}$ then follows by application of the Plancherel theorem as noted earlier.

For the simple wave equation dealt with here it seems unlikely that the results presented here will have any great bearing on ap-

plications. But these results will continue to be true, in slight-
ly modified form, for any system having a Riesz basis of eigenvec-
tors for which the corresponding eigenvalues λ_k are distributed

with uniform density in some strip of the complex plane parallel
to the imaginary axis. In particular, the results proved here can
also be demonstrated to obtain in the case of boundary control of
the Timoshenko beam ([17]) with output consisting of the measured
velocity at the point where the lateral boundary force is applied.
We see then that great care must be taken in dealing with conserva-
tive systems in regard to drawing conclusions about the input - out-
put relations applying to the continuous model on the basis of exp-
erience with finite dimensional approximations.

References

[1] Adams, R. A.: *"Sobolev Spaces"*, Academic Press, New York, San Francisco, London, 1975.

[2] Callier, F. M., and C. A. Desoer: *An algebra of transfer functions for distributed linear time - invariant systems*, IEEE CAS-25 (1978), pp. 651 - 661. See also CAS-26 (1979), p. 360, and CAS-27 (1980), pp. 320 - 323.

[3] Callier, F. M., and J. Winkin: *On spectral factorization and LQ - optimal regulation for multivariable distributed systems*, Report 88/9 of *Publications du Département de Mathematique, Facultés Universitaires de Namur*, Namur, Belgium

[4] Curtain, R. F.: *A synthesis of time and frequency domain methods for the control of infinite - dimensional systems: a system theoretic approach.* To appear in *Frontiers of Applied Mathematics*, H. T. Banks, Ed., SIAM Publications, Philadelphia.

[5] Curtain, R. F., and A. J. Pritchard: *"Infinite Dimensional Linear Systems Theory"*, Vol. 8 of *Lecture Notes in Control and Information Sciences*, Springer - Verlag, Berlin, Heidelberg, New York, 1978.

[6] Doyle, J. C., and G. Stein: *Multivariable feedback design: concepts for a classical / modern synthesis*, IEEE Trans. on Auto. Contr., 43 (1986), pp. 4 - 16.

[7] Duffin, R. J. and A. C. Schaeffer: *A class of nonharmonic Fourier series*, Tr. Am. Math. Soc., 72 (1952), pp. 341-366.

[8] Dunford, N., and J. T. Schwartz: *"Linear Operators; Part I: General Theory"*, Interscience Pub. Co., New York, 1958.

[9] Fattorini, H. O.: *Boundary control systems*, SIAM Jour. on Control, 6 (1968), pp. 349 - 388.

[10] Francis, B. A.: *"A Course in H^∞ Control Theory"*, Vol. 88 of *Lecture Notes in Control and Information Sciences*, Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[11] Ho, L. F., and D. L. Russell: *Admissible input elements for systems in Hilbert space and a Carleson measure criterion*, SIAM J. on Cont. and Opt., 21 (1983), pp. 614 - 640

[12] Kato, T.: *"Perturbation Theory for Linear Operators"*, Vol. 132 of *die Grundlehren der Mathematischen Wissen- schaften in Einzeldarstellungen*, Springer - Verlag, New York, 1966.

[13] Lax, P. D., and R. D. Richtmeyer: *Survey of the stability of linear finite difference equations*, Comm. Pure Appl. Math. 9 (1956), pp. 267 ff.

[14] Lumer, G., and R. S. Phillips: *Dissipative operators in a Banach space*, Pac. Jour. Math., 11 (1961), pp. 679 - 698.

[15] Pritchard, A. J., and D. Salamon: *The linear quadratic op- timal control problem for infinite dimensional systems with unbounded input and output operators*, SIAM J. Cont. Opt., 25 (1987), pp. 121- 144.

[16] Rabinowitz, P. H.: *Periodic solutions of nonlinear hyper- bolic partial differential equations*, Comm. Pur. Appl. Math., 20 (1967), pp. 145 - 204.

[17] Russell, D. L.: *Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions*, SIAM Review, 4 (1978), pp. 639 - 739. (See p. 669 ff.)

[18] Russell, D. L.: *On the positive square root of the fourth derivative operator*, to appear in *Quart. Appl. Math.*

[19] Salamon, D.: *Infinite - dimensional linear systems with unbounded control and observation: a functional analytic approach*, Tr. Amer. Math. Soc., 300 (1987), pp. 383 - 431.

[20] Schwartz, L.: *"Mathematics for the Physical Sciences"*, Hermann, Paris, 1966. Dist. by Addison - Wesley Pub. Co.

[21] Trotter, H. F.: *Approximation of semigroups of operators,* Pac. J. Math. 8 (1958), pp. 887 - 919.

[22] Weiss, G.: *Well-posedness, stability and asymptotic behavior of infinite dimensional linear systems,* Thesis, Weizmann Institute of Science, Rehovot, Israel, August, 1988

[23] Whittaker, E. T., and G. N. Watson: *"A Course of Modern Analysis"*, Fourth Edition, Cambridge University Press, London, 1952

[24] Wonham, W. M.: *"Linear Multivariable Control; a Geometric Approach"*, Vol. 101 of *Lecture Notes in Economics and Mathematical Systems*, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[25] Young, R. M.: *"Introduction to Nonharmonic Fourier Series"*, Academic Press, New York, 1980.